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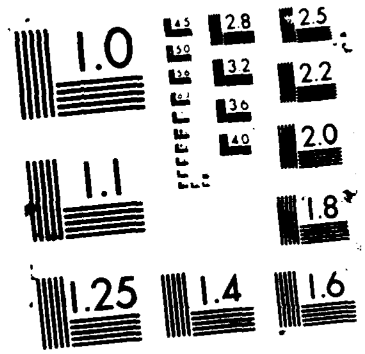
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Abstract

For a second order nonautonomous singularly perturbed ordinary differential equation with Neumann boundary conditions, the existence of single transition layer solutions is proved by using the method of Liapunov-Schmidt. The method also gives the stability of these solutions as an equilibrium point of a parabolic equation.

1. Introduction

For $\epsilon > 0$ a small parameter, we consider the following parabolic equation

(1.1) $$\dot{u} = \epsilon^2 u'' + f(u,x) \qquad -1 < x < 1,\ t > 0$$

where $\dot{u} = \partial u/\partial t$, $u' = \partial u/\partial x$ and impose the following boundary conditions

(1.2) $$u'(-1,t) = u'(1,t) = 0$$

The function f will satisfy the following assumptions:

(A-1) $f : R \times [-1,1] \to R$ is a C^∞-function of (u,x) with $f(0,x) = 0$, $f(1,x) = 0$

(A-2) *There is a positive constant* β *such that*

$$f_u(0,x),\ f_u(1,x) \leq -3\beta^2 \quad \text{for} \quad x \in [-1,1]$$

(A-3) *Let* $J(x) = \int_0^1 f(u,x)du$, $x \in [-1,1]$. *Then* $J(0) = 0$,

$$\frac{d}{dx} J(x)\Big|_{x=0} \neq 0 \quad \text{and}$$

$$\int_0^u f(v,0)dv < 0 \quad \text{for} \quad u \in (0,1).$$

An example of a function f satisfying (A-1) - (A-3) is the cubic

(1.3) $$f(u,x) = u(1 - u)(u - a(x))$$

where

(1.4) $$a(0) = \tfrac{1}{2},\ a'(0) \neq 0,\ 0 < a(x) < 1 \quad \text{for} \quad x \in [-1,1]$$

We intend to discuss the existence of equilibrium solutions of (1.1), (1.2) with a single transition layer at $x = 0$; that is, those solutions which, as

$\epsilon \to 0$, converge to zero (resp., one) uniformly on any compact subset of $[-1,0)$ and converge to one (resp., zero) uniformly on any compact set of $(0,1]$. We also discuss the stability properties of such solutions.

In fact, we prove the following theorem.

Theorem 1.1. (i) *There is an* $\epsilon_0 > 0$ *and two single transition layer solutions* $u_j(x,\epsilon)$, $j = 0,1$, $0 < \epsilon \leq \epsilon_0$, *such that, for any* $\delta > 0$,

$$\lim_{\epsilon \to 0} u_j(x,\epsilon) = j \quad \textit{uniformly on} \quad [-1,-\delta],$$

$u_j(x,\epsilon)$ *is asymptotically stable if*

$$(1.5) \qquad (-1)^j u_j'(0,\epsilon) J'(0) > 0$$

and unstale with the dimension of the unstable manifold equal to one if $(-1)^j u_j'(0,\epsilon) J'(0) < 0$.

Before describing the method of proof, let us first emphasize that this result is certainly not surprising and is probably known to some people. In fact, the existence of equilibrium solutions of the above type follows from the work of Fife [1976], Ito [1984], Mimura, Tabata and Hosono [1980]. The method employed there is to reflect the solution through -1 and +1, solve two distinct boundary value problems on the intervals $[-2,0]$ and $[0,2]$ and then use the boundary condition at zero to match the derivatives of the solutions.

For the case of the cubic (1.3), (1.4), Angenent, Mallet-Paret and Peletier [1987] have obtained the stability condition (1.5). They obtained existence using a comparison principle and results of Matano [10] on existence of stable solutions. Some of the techniques used there are of assistance to us in discussing

all solutions for the general case. One could also obtain the stability properties of the solutions by using the method of Fujii and Nisihura [1985] involving a singular limit eigenvalue problem.

The primary objective of this paper is to prove this theorem by using a method which will yield the existence and stability at the same time. More specifically, we begin with a smooth approximate equilibrium solution $U(x,\epsilon)$ of the equation which exhibits a transition layer at $x = 0$ and then consider the dynamics of the flow in a neighborhood of this approximate solution. The variational equation near this approximate solution has the form

$$u_t = \mathfrak{L}^\epsilon u + G(\epsilon) + F(u,\epsilon) \tag{1.6}$$

where

$$\mathfrak{L}^\epsilon u = \epsilon^2 u'' + f_u(U(x,\epsilon),x)u \tag{1.7}$$

$$G(\epsilon)(x) = \epsilon^2 U''(x,\epsilon) + f(U(x,\epsilon),x) \tag{1.8}$$

$$F(u,\epsilon) = f(U(x,\epsilon) + u,x) - f(U(x,\epsilon),x) - f_u(U(x,\epsilon),x)u \tag{1.9}$$

The first step in any analysis of Equation (1.6) must involve an understanding of the operator $\mathfrak{L}^\epsilon : C^2_\epsilon[-1,1] \to C^0[-1,1]$ where $C^2_\epsilon[-1,1]$ is the space of C^2-functions satisfying the boundary conditions (1.2) with $|\varphi|_{2,\epsilon} = \sup_x[|\varphi(x)| + \epsilon|\varphi'(x)| + \epsilon^2|\varphi''(x)|]$. By using a Prufer transformation and analyzing the behavior of the corresponding angle, we show there is exactly one eigenvalue $\lambda_1(\epsilon)$ of $\mathfrak{L}^\epsilon$ which approaches zero as $\epsilon \to 0$ and $\lambda_1'(0)$ is proportional to $J'(0)$. Furthermore, there is an $\epsilon_0 > 0$, $\nu > 0$ such that the remaining eigenvalues are $\leq -\nu$ for $0 < \epsilon \leq \epsilon_0$. The use of the Prufer transformation in the study of stability of solutions of parabolic equations has been used previously by Fusco and Hale [1985], Hale and Rocha [1985], Jones [1984], and Rocha [1985], [1986].

After obtaining this information about $\mathfrak{L}^\epsilon$, two approaches naturally suggest themselves. One is to use a center manifold theorem to reduce the dynamics near the approximate solution to a one dimensional problem. The other is to use the method of Liapunov-Schmidt to obtain a one dimensional bifurcation function whose zeros determine the equilibrium solutions. In some situations for which it is known that both of these methods can be applied, the flow of the vector field defined by the bifurcation function is equivalent to the flow on the center manifold (see, for example, Chow and Hale [1982]).

In this paper, we consider the method of Liapunov-Schmidt for the existence of the equilibrium solutions. The stability properties of the solutions are obtained by discussing the eigenvalues of the linear variational equation directly. The existence of the center manifold and its relationship to the bifurcation function will appear in a later publication.

To apply the method of Liapunov-Schmidt, the accuracy of the initial approximation $U(x,\epsilon)$ plays a crucial role. To see this, let $\varphi_1(x,\epsilon)$ be an eigenfunction of $\mathfrak{L}^\epsilon$ corresponding to $\lambda_1(\epsilon)$, and consider the equation for equilibrium solutions

$$\mathfrak{L}^\epsilon u + G(\epsilon) + F(u,\epsilon) = 0.$$

If $u = \alpha\varphi_1 + v$ where $\int_{-1}^1 \varphi_1(x,\epsilon)v(x)dx = 0$ and α is a scalar, then the method of Liapunov-Schmidt yields a function $v^*(\alpha,\epsilon)$ defined for α,ϵ small, $v^*(0,0) = 0$. Once $v^*(\alpha,\epsilon)$ is known, the bifurcation function is given by

$$B(\alpha,\epsilon) = \lambda_1(\epsilon)\alpha + \int_{-1}^1 \varphi_1(x,\epsilon)\Big[G(\epsilon)(x) + F(\alpha\varphi_1(x,\epsilon) + v^*(\alpha,\epsilon)(x),\epsilon)\Big]dx/\|\varphi_1(\epsilon)\|_{L^2}^2$$

The desired transition layer solutions are in one-to-one correspondence with the zeros of $B(\alpha,\epsilon)$.

The Taylor series for $B(\alpha,\epsilon)$ will have the form

$$B(\alpha,\epsilon) = \beta(\epsilon) + \gamma(\epsilon)\alpha + O(\alpha^2)$$

as $\alpha \to 0$. The Taylor series of the terms $\beta(\epsilon)$, $\gamma(\epsilon)$ in ϵ depend very strongly upon the initial approximation $U(x,\epsilon)$ to the equilibrium solution of (1.1), (1.2); that is, upon the properties of the function $G(\epsilon)$ in (1.8). More specifically, suppose $G(\epsilon)$ is only $O(\epsilon)$ as $\epsilon \to 0$ and

$$\int_{-1}^{1} \varphi_1(x,\epsilon)G(\epsilon)(x)dx = \beta_0\epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

with $\beta_0 \neq 0$. The function $v^*(\alpha,\epsilon)$ then will satisfy $v^*(\alpha,\epsilon) = O(\epsilon + \alpha^2)$ as $\epsilon,\alpha \to 0$ and

$$B(\alpha,\epsilon) = [\beta_0\epsilon + o(\epsilon)] + [\gamma_0\epsilon + o(\epsilon)]\alpha + O(\alpha^2)$$

where γ_0 is determined from the first eigenvalue of $\mathfrak{L}^\epsilon$ and the $O(\epsilon)$ term in $v^*(0,\epsilon)$. In this case, the equation $B(\alpha,\epsilon) = 0$ will not have a solution $\alpha^*(\epsilon)$ which vanishes when $\epsilon = 0$. Thus, there is no equilibrium solution of (1.1), (1.2) which is a perturbation of $U(x,\epsilon)$ in the direction of the eigenfunction $\varphi_1(x,\epsilon)$.

If one wants to obtain existence of the equilibrium solutin (as well as its stability) by perturbing an approximation solution in the direction of the eigenfunction $\varphi_1(x,\epsilon)$, then the above reasoning implies that the initial approximation must be more accurate. If we suppose that $U(x,\epsilon)$ is such that $G(\epsilon)$ is $O(\epsilon^2)$ as $\epsilon \to 0$, then $v^*(\alpha,\epsilon) = O(\epsilon^2 + \alpha^2)$ and the bifurcation function $B(\alpha,\epsilon)$ has $\beta(\epsilon) = O(\epsilon^2)$ and $\gamma(\epsilon) = \gamma_0(\epsilon) + O(\epsilon^2)$ as $\epsilon \to 0$, $\gamma_0'(0) \neq 0$. This implies that, if $B(\alpha,\epsilon) = 0$, then $\alpha = O(\epsilon)$ as $\epsilon \to 0$, and there is an exact equilibrium solution near $U(x,\epsilon)$ which is a perturbation in the direction of the first eigenfunction $\varphi_1(x,\epsilon)$.

Sections 2,3, and 4 are devoted respectively to the discussion of the approximate solution, the linear operator $\mathfrak{L}^{\epsilon}$ and the application of the method of Liapunov-Schmidt.

It is possible that the function $J(x)$ in (A-3) could have more than one zero. In this case, solutions with several transition layers may occur, a situation which is discussed in Section 5. It is also possible that the function $f(u,x)$ does not have zeros which are constant in x as in hypothesis (A-1). The modifications that are necessary to handle this case are discussed in Section 6.

2. An Approximation

In this section, we describe a way to obtain an approximation to the equilibrium solutions of (1.1), (1.2) with a single transition layer under the hypotheses (A-1) - (A-3). If $u(x;\epsilon)$ is an equilibrium solution and we let $z(t,\epsilon) = u(\epsilon t,\epsilon)$, "$\cdot$" = d/dt, then

(2.1) $$\ddot{z} + f(z,\epsilon t) = 0$$

on $-^1/\epsilon < t < {}^1/\epsilon$ with the boundatry conditions $\dot{z} = 0$ at $t = \pm\epsilon^{-1}$. To obtain the approximate solution, we let

(2.2) $$z(t,\epsilon) = z_0(t) + \epsilon z_1(t) + O(\epsilon^2)$$

and formally equate powers of ϵ in (2.1), then

(2.3) $$\ddot{z}_0 + f(z_0,0) = 0$$

(2.4) $$\ddot{z}_1 + f_u(z_0(t),0)z_1 + f_x(z_0(t)),0)t = 0$$

for $t \in \mathbb{R}$ The boundary conditions are

(2.5) $$z_0(-\infty) = 0, \; z_0(+\infty) = 1,$$

(2.6) $$z_1(\pm\infty) = 0 \; .$$

The function z_0 will give a solution with transition from 0 to 1. For the case of a transition from 1 to 0, one imposes the conditions $z_0(-\infty) = 1$, $z_0(+\infty) = 0$ instead of (2.5).

The conditions (A-1) and (A-2) imply that Equation (2.3) has equilibrium points (0,0), (1,0) in the $(z_0,\dot{z}_0)$ phase plane which are hyperbolic saddle points. Furthermore, Condtion (A-3) implies that there is a heteroclinic orbit $(z_0(t,\gamma), \dot{z}_0(t,\gamma))$ which connects the equilibrium point (0,0) to the equilibrium point (1,0). The constant $\gamma \in (0,1)$ is the initial value of $z_0(t;\gamma)$: that is, $z_0(0,\gamma) = \gamma$, and uniquely specifies the heteroclinic orbit. Moreover, there is a constant $k_0 > 0$ such that

$$\max\{|z_0(t,\gamma)-1|, |\dot z_0(t,\gamma)|\} \le k_0 e^{-2\beta t},\ t \ge 0 \tag{2.7}$$
$$\max\{|z_0(t,0)|, |\dot z_0(t,\gamma)|\} \le k_0 e^{2\beta t},\ t \le 0.$$

With this choice of $z_0(t,\gamma)$, one can now begin to discuss a solution $z_1(t)$ of (2.4) which satisfies the boundary condition (2.6). The linear equation

$$\ddot z_1 + f_u(z_0(t),0)z_1 = 0$$

has the property that the only bounded solution on $\mathbb{R}$ is a multiple of $\dot z_0(t,\gamma)$. Using a well known theory based on exponential dichotomy and the Fredholm alternative (see, for example, Chow and Hale, [1982, Sec. 11.3], Hale [1984, pp. 123]), the equation (2.4) has a solution which is bounded on $\mathbb{R}$ if and only if

$$\int_{-\infty}^{\infty} \dot z_0(t,\gamma) f_x(z_0(t,\gamma),0)t\,dt = 0. \tag{2.8}$$

Therefore, condition (2.8) must be satisfied in order to obtain a solution of (2.4) satisfying (2.6).

We now show that there is a unique $\gamma \in (0,1)$ such that (2.8) is satisfied. To see this, let

$$C_0(\gamma) = \int_{-\infty}^{\infty} \dot z_0(t,\gamma) f_x(z_0(t,\gamma),0)t\,dt = \int_0^1 f_x(u,0)\left(\int_\gamma^u [-2F(v)]^{-1/2} dv\right)du$$

in which $F(u) = \int_0^u f(s,0)ds$. Since

$$\frac{d}{d\gamma} C_0(\gamma) = -[-2F(\gamma)]^{-1/2} \int_0^1 f_x(u,0)du = -[-2F(\gamma)]^{-1/2} J'(0) \ne 0$$

it follows that $C_0(\gamma)$ is strictly monotone. On the other hand, $F(\gamma) = O(\gamma^2)$ as $\gamma \to 0$ and $F(\gamma) = O((\gamma-1)^2)$ as $\gamma \to 1$. Therefore, $|C_0(\gamma)| \to \infty$ as $\gamma \to 0$ or 1 and there is a unique γ in (0,1) such that (2.8) holds.

Let us choose γ so that (2.8) holds and now designate $z_0(t,\gamma)$ by $z_0(t)$.

Then there is a solution of (2.4) bounded on $\mathbb{R}$ In fact, there is a solution which is a continuous linear functional in the uniform topology on $\mathbb{R}$ of the forcing function $f_x(z_0(t),0)t$ in (2.4). Since $z_0(t)$ satisfies the estimate (2.7) and the forcing term satisfies

$$|f_x(z_0(t),0)t| \leq k_0 e^{-2\beta|t|} \quad t \in \mathbb{R}$$

it follows that $z_1(t)$ satisfies the estimate

(2.9) $$\max\{|z_1(t)|, |\dot{z}_1(t)|\} \leq k_1 e^{-2\beta|t|}, \; t \in \mathbb{R}$$

for some positive constant k_1.

Now, let $\zeta_0(x)$, $\zeta_+(x)$ be C^∞-cutoff functions satisfying

$$\zeta_0(x) = \begin{cases} 1 & |x| \leq 1/4 \\ 0 & |x| \geq 1/2 \\ 0 \leq \zeta_0(x) \leq 1, & x \in [-1,1]. \end{cases}$$

$$\zeta_+(x) \quad \begin{cases} 0 & x \in [-1,0] \\ 1 - \zeta_0(x) & x \in [0,1]. \end{cases}$$

and let

(2.10) $$Z(t,\epsilon) = z_0(t) + \epsilon z_1(t)$$

(2.11) $$U(x,\epsilon) = \zeta_0(x) Z(x/\epsilon,\epsilon) + \zeta_+(x).$$

The function U *will be our approximation to the equilibrium solution of* (1.1), (1.2). If we make the transformation of variables,

(2.12) $$u \longmapsto U(x,\epsilon) + u$$

in (1.1), then the new function u must satisfy the differential equation

(2.13) $$\frac{\partial u}{\partial t} = \mathcal{L}^\epsilon u + G(\epsilon) + F(u,\epsilon)$$

and the boundary conditions (1.2), where

$$\mathcal{L}^{\epsilon}u = \epsilon^2 u'' + f_u(U(x,\epsilon),x)u \tag{2.14}$$

$$G(\epsilon)(x) = \epsilon^2 U''(x,\epsilon) + f(U(x,\epsilon),x) \tag{2.15}$$

$$F(u,\epsilon)(x) = f(U(x,\epsilon) + u,x) - f(U(x,\epsilon),x) - f_u(U(x,\epsilon),x)u. \tag{2.16}$$

As is seen from (2.15), $G(\epsilon)$ measures how accurately $U(x,\epsilon)$ approximates the equilibrium solution of (1.1), (1.2). The following lemma partly justifies the expansion in (2.2).

Lemma 2.1. $\sup_{x\in[-1,1]} |G(\epsilon)(x)| = O(\epsilon^2)$ as $\epsilon \to 0$.

Proof. Let us first write $G(\epsilon)(x)$ explicitly:

$$\begin{aligned} G(\epsilon)(x) &= \epsilon^2 U''(x,\epsilon) + f(U(x,\epsilon),x) \\ &= \zeta_0(x)[\ddot{z}_0(x/\epsilon) + \epsilon\ddot{z}_1(x/\epsilon)] \\ &\quad + 2\epsilon\zeta_0'(x)[\dot{z}_0(x/\epsilon) + \epsilon\dot{z}_1(x/\epsilon)] \\ &\quad + \epsilon^2\zeta_0''(x)[z_0(x/\epsilon) + \epsilon z_1(x/\epsilon)] + \epsilon^2\zeta_+''(x) \\ &\quad + f(\zeta_0(x)[z_0(x/\epsilon) + \epsilon z_1(x/\epsilon)] + \zeta_+(x),x). \end{aligned}$$

From the choice of the cut-off functions ζ_0, ζ_+, one easily obtains the following estimates:

a) $|x| \geq {}^1/_2$

$$|G(\epsilon)(x)| = |f(\zeta_+(x),x)| = 0$$

b) $-{}^1/_2 \leq x \leq -{}^1/_4$

$$\begin{aligned} |G(\epsilon)(x)| &\leq \sup\{|\ddot{z}_0(x/\epsilon)| + \epsilon|\ddot{z}_1(x/\epsilon)|;\ x \in [-{}^1/_2, -{}^1/_4]\} \\ &\quad + 2\epsilon|\zeta_0'|_0 \sup\{|\dot{z}_0(x/\epsilon) + \epsilon/\dot{z}_1(x/\epsilon)|;\ x \in [-{}^1/_2, -{}^1/_4]\} \\ &\quad + \epsilon^2|\zeta_0''|_0 \sup\{|z_0(x/\epsilon)| + \epsilon|z_1(x/\epsilon)|;\ x \in [-{}^1/_2, -{}^1/_4]\} \\ &\leq c\, e^{-\beta/2\epsilon} \end{aligned}$$

in which c is a positive constant independent of x and $\epsilon > 0$.

c) $\quad {}^1/_4 \leq x \leq {}^1/_2$

$$\begin{aligned} |G(\epsilon)(x)| &\leq \sup\{|\ddot z_0(x/\epsilon)| + \epsilon|\ddot z_1(x/\epsilon)|;\ x \in [{}^1/_4, {}^1/_2]\} \\ &\quad + 2\epsilon|\zeta_0'|_0 \sup\{|\dot z_0(x/\epsilon)| + \epsilon|\dot z_1(x/\epsilon)|;\ x \in [{}^1/_4, {}^1/_2]\} \\ &\quad + \epsilon^2|\zeta_0''|_0 \sup\{|z_0(x/\epsilon) - 1| + \epsilon|z_1(x/\epsilon)|;\ x \in [{}^1/_4, {}^1/_2]\} \\ &\leq c\, e^{-\beta/2\epsilon} \end{aligned}$$

d) $\quad |x| \leq {}^1/_4$

$$\begin{aligned} G(\epsilon)(x) &= f(z_0(x/\epsilon) + \epsilon z_1(x/\epsilon),x) - f(z_0(x/\epsilon), 0) \\ &\quad -\epsilon[f_u(z_0(x/\epsilon),0)z_1(x/\epsilon) + f_x(z_0(x/\epsilon),0)x/\epsilon] \end{aligned}$$

By applying the mean value theorem, one finds a $\theta = \theta(x,\epsilon)$, $0 \leq \theta(x,\epsilon) \leq 1$, such that, for $|x| \leq {}^1/_4$

$$\begin{aligned} G(\epsilon)(x) &= \tfrac{1}{2}f_{uu}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)[\epsilon z_1(x/\epsilon)]^2 \\ &\quad + f_{ux}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)xz_1(x/\epsilon)\epsilon \\ &\quad + \tfrac{1}{2}f_{xx}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)x^2 \\ &= \epsilon^2[\tfrac{1}{2} f_{uu}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)z_1(x/\epsilon)^2 \\ &\quad + f_{ux}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)\, \tfrac{x}{\epsilon}\, z_1(x/\epsilon) \\ &\quad + \tfrac{1}{2} f_{xx}(z_0(x/\epsilon) + \theta\epsilon z_1(x/\epsilon),\theta x)(\tfrac{x}{\epsilon})^2] \end{aligned} \tag{2.17}$$

Since $z_0(t)$, $z_1(t)$ satisfy the estimates (2.7), (2.9) and $f_{xx}(0,x) \equiv 0 \equiv f_{xx}(1,x)$, the function

$$\begin{aligned} &\tfrac{1}{2} f_{uu}(z_0(t) + \epsilon\theta z_1(t),\ \theta\epsilon t)z_1(t)^2 \\ &\quad + f_{ux}(z_0(t) + \epsilon\theta z_1(t),\ \theta\epsilon t)tz_1(t) \\ &\quad + \tfrac{1}{2} f_{xx}(z_0(t) + \epsilon\theta z_1(t),\ \theta\epsilon t)t^2 \end{aligned}$$

is bounded on $\mathbb{R}$ as a function of t, which together with (2.17) implies

$$\sup_{x\in[-{}^1/_4,{}^1/_4]} |G(\epsilon)(x)| \leq c\ \epsilon^2.$$

Since $\lim_{\epsilon \downarrow 0} \epsilon^{-2} e^{-\beta/2\epsilon} = 0$, the estimates in a) ~ d) imply

$$\sup_{|x| \leq 1} |G(\epsilon)(x)| \leq C\epsilon^2 \quad \text{as} \quad \epsilon \downarrow 0$$

for some positive constant C. The proof is complete.

3. Properties of the linear operator

In this section, we intend to discuss the spectral properties of the linear operator $\mathfrak{L}^\epsilon$ in (2.14). Let

$$X = \{u \in C^2[-1,1];\ u'(-1) = 0 = u'(1)\}$$

$$Y = C^0[-1,1].$$

and

$$|u|_{2,\epsilon} = |u|_0 + \epsilon|u'|_0 + \epsilon^2|u''|_0 \text{ for } u \in X.$$

We consider $\mathfrak{L}^\epsilon : X \to Y$ and prove the following

Theorem 3.1. *There is an* $\epsilon_0 > 0$ *such that the following assertions are valid:*

(i) *The principal eigenvalue* $\lambda_1(\epsilon)$ *of* $\mathfrak{L}^\epsilon$ *is simple and approaches zero as* $\epsilon \downarrow 0$.

(ii) *If* $\phi(x,\epsilon)$ *is any eigenfunction corresponding to* $\lambda_1(\epsilon)$, $0 < \epsilon \leq \epsilon_0$, *then there is a constant* $k_2 > 0$, *such that*

$$|\phi(x,\epsilon)| \leq k_2\ |\phi(0,\epsilon)|\ e^{-2\beta|x|/\epsilon} \quad \text{for} \quad |x| \leq 1.$$

(iii) *There is a* $\mu_0 > 0$ *such that, the second eigenvalue* $\lambda_2(\epsilon)$ *of* $\mathfrak{L}^\epsilon$ *satisfies* $\lambda_2(\epsilon) \leq -\mu_0$ *for* $0 < \epsilon \leq \epsilon_0$.

To prove this result, we first observe that the eigenvalue problem

$$\mathfrak{L}^\epsilon u = \lambda u \tag{3.1}$$

is equivalent to the system of first order equations

$$(3.2)_\lambda \qquad \begin{cases} \epsilon u' = u & x \in [-1,1] \\ \epsilon v' = -[f_u(U(x;\epsilon),x) - \lambda]u & \\ v(-1) = 0 = v(1). & \end{cases}$$

In terms of the fast variable $t = x/\epsilon$, $(3.2)_\lambda$ is written as

$$(3.3)_\lambda \qquad \begin{cases} \dot{u} = v & t \in [-{}^1/\epsilon, {}^1/\epsilon] \\ \dot{u} = -[f_u(\tilde{U}(t,\epsilon),\epsilon t) - \lambda]u & \\ u(-{}^1/\epsilon) = 0 = u({}^1/\epsilon) & \end{cases}$$

in which $\tilde{U}(t,\epsilon) = U(\epsilon t,\epsilon)$. It is useful to introduce polar coordinates (r,θ) defined by $u = r\cos\theta$, $v = -r\sin\theta$, in $(3.2)_\lambda$ and $(3.3)_\lambda$, and determine the properties of the eigenvalues of $\mathfrak{L}^\epsilon$ from the properties of the angle θ.

The equations for (r,θ) are given by

$$(3.4)_\lambda \qquad \begin{array}{ll} \text{a)} & \epsilon r' = -r[1 + \lambda - f_u(U(x,\epsilon),x)]\sin\theta\cos\theta \\ \text{b)} & \epsilon\theta' = [f_u(U(x,\epsilon),x) - \lambda]\cos^2\theta + \sin^2\theta \end{array}$$

and

$$(3.5)_\lambda \qquad \begin{array}{ll} \text{a)} & \dot{r} = -[1 + \lambda - f_u(\tilde{U}(t,\epsilon), \epsilon t)]\sin\theta\cos\theta \\ \text{b)} & \dot{\theta} = [f_u(\tilde{U}(t,\epsilon),\epsilon t) - \lambda]\cos^2\theta + \sin^2\theta. \end{array}$$

Let $\theta_\pm(x,\epsilon,\lambda)$ denote a unique solution of $(3.4\text{-b})_\lambda$ with $\theta_\pm(\pm 1,\epsilon,\lambda) = 0$. Then it easily follows that λ is an eigenvalue of $\mathfrak{L}^\epsilon$ if and only if $\theta_-(1,\epsilon,\lambda) \equiv 0(\mathrm{mod.}\pi)$ (or $\theta_+(-1,\epsilon,\lambda) \equiv 0(\mathrm{mod.}\pi)$).

Moreover, one can verify that $\theta_-(1,\epsilon,\lambda)$ (resp. $\theta_+(-1,\epsilon,\lambda)$) is a strictly decreasing (resp. increasing) function of λ for each fixed $\epsilon > 0$ and $\lim_{\lambda\to-\infty}\theta_-(1,\epsilon,\lambda) = \infty$, $\lim_{\lambda\to+\infty}\theta_-(1,\epsilon,\lambda) = -{}^\pi/2$. Therefore, the principal eigenvalue $\lambda_1(\epsilon)$ of $\mathfrak{L}^\epsilon$ is characterized by:

$$\theta_-(1,\epsilon,\lambda_1(\epsilon)) = 0.$$

There is a constant $\delta > 0$ such that $\{|f_u(u,x) - f_u(0,x)|, |f_u(1 + u,x) - f(1,x)|\} \leq \beta^2/2$ for $|u| \leq \delta$ and from (2.7), (2.9) and (2.11), it follows that there exists a constant $k_2 > 0$ such that

$$|U(x,\epsilon)| \leq k_2(1+\epsilon)\, e^{-\beta|x|/2\epsilon} \quad \text{for} \quad x \leq 0$$

$$|U(x,\epsilon) - 1| \leq k_2(1+\epsilon)\, e^{-\beta x/2\epsilon} \quad \text{for} \quad x \geq 0.$$

From the condition (A-2) and the estimates above, it follows that, for any $\bar{t}_0$, $\bar{\epsilon}_0$ satisfying the relation

$$\text{(3.6)} \qquad k_2(1+\bar{\epsilon}_0)\, e^{-\beta\bar{t}_0/2} < \delta.$$

and for any $t_0 \geq \bar{t}_0$, $0 < \epsilon \leq \bar{\epsilon}_0$, one has

$$\text{(3.7)} \qquad f_u(U(x,\epsilon),x) - \lambda \leq -2\beta^2 \quad \text{for} \quad |x| \geq \epsilon t_0,\ |\lambda| < \beta^2/2.$$

Now let $\Theta_\pm(x,\epsilon,\lambda)$ be solutions of the equation

$$[f_u(U(x,\epsilon),x) - \lambda] \cos^2 \Theta + \sin^2 \Theta = 0$$

for $|x| \geq \epsilon t_0$, $|\lambda| \leq \beta^2/2$, satisfying

$$-\pi/2 < \Theta_-(x,\epsilon,\lambda) \leq -\Theta_0$$

$$\Theta_0 \leq \Theta_+(x,\epsilon,\lambda) < \pi/2$$

in which $\Theta_0 \in (0,\pi/2)$ is the unique solution of

$$\tan^2 \Theta_0 = 2\beta^2.$$

Lemma 3.2. *Suppose* $\bar{\epsilon}_0$, $\bar{t}_0$ *satisfy* (3.6). *Then there exist constants* ϵ_1, $0 < \epsilon_1 \leq \bar{\epsilon}_0$, $k_3 > 0$, $k_4 > 0$, $\bar{B} > 0$ *and solutions* $\bar{\Theta}_\pm(x,\epsilon,\lambda)$ *of* $(3.4\text{-b})_\lambda$ *such that, for* $\epsilon \in (0,\epsilon_1]$, $t_0 \geq \bar{t}_0$, *and* $\lambda \in [-\beta^2/2, \beta^2/2]$, *the following properties hold:*

(i) $$|\bar{\Theta}_+(x,\epsilon,\lambda) - \Theta_+(x,\epsilon,\lambda)| \leq k_3\, \epsilon, \quad \text{for} \quad |x| \geq \epsilon t_0$$

$$|\bar{\Theta}_-(x,\epsilon,\lambda) - \Theta_-(x,\epsilon,\lambda)| \leq k_3\, \epsilon, \quad \text{for} \quad |x| \geq \epsilon t_0$$

(ii) *If* $\theta(x,\epsilon,\lambda)$ *is the solution of* $(3.4\text{-b})_\lambda$ *with initial value* $\theta(-1,\epsilon,\lambda)$ (resp. $\theta(+1,\epsilon,\lambda)$) *satisfying*

$$\theta(-1,\epsilon,\lambda) \in (\Theta_+(-1,\epsilon,\lambda) + k_3\epsilon + (n-1)\pi, \Theta_+(-1,\epsilon,\lambda) - k_3\epsilon + n\pi)$$

$$\text{(resp.} \quad \theta(1,\epsilon,\lambda) \in (\Theta_-(1,\epsilon,\lambda) + k_3\epsilon + n\pi,\ \Theta_-(1,\epsilon,\lambda) - k_3\epsilon + (n+1)\pi))$$

for some integer n, *then the following estimate is valid:*

$$|\theta(x,\epsilon,\lambda) - \overline{\theta}_-(x,\epsilon,\lambda) - n\pi| \leq k_4 e^{-2\overline{\beta}(x+1)/\epsilon} \quad \text{for} \quad x \in [-1,-\epsilon t_0]$$

$$(\text{resp.} \quad |\theta(x,\epsilon,\lambda) - \overline{\theta}_+(x,\epsilon,\lambda) - n\pi| \leq k_4 e^{2\overline{\beta}(x-1)/\epsilon} \quad \text{for} \quad x \in [\epsilon t_0,1])$$

(iii) *If* $\theta(x,\epsilon,\lambda)$ is the solution of $(3.4\text{-b})_\lambda$ with the initial value $\theta(-\epsilon t_0,\epsilon,\lambda)$ (resp. $\theta(\epsilon t_0,\epsilon\lambda)$) at $x = -\epsilon t_0$ (resp. $x = \epsilon t_0$) satisfying

$$\theta(-\epsilon t_0,\epsilon,\lambda) \in (\Theta_-(-\epsilon t_0,\epsilon,\lambda) + k_3\epsilon + n\pi,\ \Theta_-(-\epsilon t_0,\epsilon,\lambda) - k_3\epsilon + (n+1)\pi)$$

$$(\text{resp.} \quad \theta(\epsilon t_0,\epsilon,\lambda) \in (\Theta_+(\epsilon t_0,\epsilon,\lambda) + k_3\epsilon + (n-1)\pi,\ \Theta_+(\epsilon t_0,\epsilon,\lambda) - k_3\epsilon + n\pi))$$

for some integer n, *then the following estimate is valid;*

$$|\theta(x,\epsilon,\lambda) - \overline{\theta}_+(x,\epsilon,\lambda) - n\pi| \leq k_4 e^{2\overline{\beta}(x+\epsilon t_0)/\epsilon}, \quad \textit{for} \quad x \in [-1,-\epsilon t_0]$$

$$(\text{resp.} \quad |\theta(x,\epsilon,\lambda) - \overline{\theta}_-(x,\epsilon,\lambda) - n\pi| \leq k_4 e^{-2\overline{\beta}(x-\epsilon t_0)/\epsilon}, \quad \textit{for} \quad x \in [\epsilon t_0,1])$$

Proof of Lemma 3.2. To prove these results it is convenient to rewrite the equation $(3.4\text{-b})_\lambda$ in terms of coordinates around $\Theta_\pm(x,\epsilon,\lambda)$. For this purpose, let us introduce new coordinates $(\overline{u},\overline{v})$ in $(3.2)_\lambda$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -A & A \end{bmatrix} \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}$$

where $A = A(x,\epsilon,\lambda) = [-f_u(U(x,\epsilon),x) + \lambda]^{1/2}$ for $|x| \geq \epsilon t_0$. One should notice that $A(x,\epsilon,\lambda) \geq \sqrt{2}\beta$ holds for $|x| \geq \epsilon t_0$, and hence the change of coordinates makes sense for $|x| \geq \epsilon t_0$. Then introduce polar coordinates (ρ,ϕ) by $\overline{u} = \rho \cos \phi$, $\overline{v} = -\rho \sin \phi$. The Equations for (ρ,ϕ) are given by

$$(3.8)_\lambda \quad \begin{aligned} &\text{a)} \quad \epsilon\rho' = -A(x,\epsilon,\lambda)\rho \cos 2\phi - \epsilon(A'/2A)\rho[1 - \sin 2\phi] \\ &\text{b)} \quad \epsilon\phi' = A(x,\epsilon,\lambda) \sin 2\phi - \epsilon(A'/2A) \cos 2\phi \end{aligned}$$

$\theta = \Theta_-(x,\epsilon,\lambda)$ and $\theta = \Theta_+(x,\epsilon,\lambda)$ are transformed into $\phi = -\pi/2$ and $\phi = 0$, respectively. The Neumann data $\theta \equiv 0(\text{mod.}\pi)$ corresponds to $\phi \equiv \pi/4\ (\text{mod.}\pi)$. The equation $(3.8\text{-b})_\lambda$ has positively (resp. negatively) invariant strips around

$\phi = \pi/2 + m\pi$, $m \in Z$ (resp. $\phi = m\pi$) for $|x| \geq \epsilon t_0$. The width of these strips is given by:

$$\tan^{-1}(\epsilon \sup_{|x| \geq \epsilon t_0} |A'(x,\epsilon,\lambda)/2A(x,\epsilon,\lambda)^2|) \leq \bar{k}_3 \epsilon$$

for some positive constant $\bar{k}_3$. Since $A(x,\epsilon,\lambda) \geq \sqrt{2}\beta$ and θ and ϕ coordinates are related by

$$\tan \theta / \tan (\phi + \pi/4) = A(x,\epsilon,\lambda), \quad |x| \geq \epsilon t_0$$

there exists a positive constant k_3, such that the interval

$$(m\pi + \Theta_-(x,\epsilon,\lambda) - k_3\epsilon,\ m\pi + \Theta_-(x,\epsilon,\lambda) + k_3\epsilon)$$

is positively invariant for the equation $(3.4\text{-b})_\lambda$ and the interval

$$(m\pi + \Theta_+(x,\epsilon,\lambda) - k_3\epsilon,\ m\pi + \Theta_+(x,\epsilon,\lambda) + k_3\epsilon)$$

is negatively invariant for the equation $(3.4\text{-b})_\lambda$ for any integer m. Now it is easy to find the solutions $\bar{\theta}_\pm(x,\epsilon,\lambda)$ of $(3.4\text{-b})_\lambda$ which satisfy the property (i) in the lemma. For example, $\bar{\theta}_-(x,\epsilon,\lambda)$ is defined as the unique solution of $(3.4\text{-b})_\lambda$ with initial value $\bar{\theta}_-(-1,\epsilon,\lambda) = \Theta_-(-1,\epsilon,\lambda)$ for $x \in [-1,-\epsilon t_0]$ and as the unique solution of $(3.4\text{-b})_\lambda$ with initial value $\bar{\theta}_-(+\epsilon t_0,\epsilon,\lambda) = \Theta_-(+\epsilon t_0,\epsilon,\lambda)$, for $x \in [+\epsilon t_0, 1]$. The function $\bar{\theta}_+$ is defined in the similar way. This completes the proof of Part (i).

Now considering the difference $\theta(x,\epsilon,\lambda) - \bar{\theta}_-(x,\epsilon,\lambda)$ and applying the mean value theorem, the equation $(3.4\text{-b})_\lambda$ yields $|\theta(x,\epsilon,\lambda) - \bar{\theta}_-(x,\epsilon,\lambda)| \leq k_4 e^{-2\bar{\beta}(x+1)/\epsilon}$ for $x \in [-1,-\epsilon t_0]$. All other statements in (ii), (iii) follow using the same type of arguments. The proof of Lemma 3.2 is complete.

Returning to the proof of Theorem 3.1, we examine the behavior of $\theta_\pm(x,\epsilon,\lambda)$ over the interval $[-\epsilon t_0, \epsilon t_0]$. Let us concentrate our attention on $\theta_\pm(x,\epsilon,0)$ for the moment. At this stage, it is convenient to use the equation $(3.5\text{-b})_0$ in terms of the fast variable $t = x/\epsilon$:

(3.9) $$\dot{\theta} = f_u(Z(t,\epsilon),\epsilon t)\cos^2\theta + \sin^2\theta, \quad t \in [-{}^1/4\epsilon, {}^1/4\epsilon].$$

The solutions of this equation are compared to those of

(3.10) $$\dot{\theta} = f_u(z_0(t),0)\cos^2\theta + \sin^2\theta, \; t \in (-\infty,\infty).$$

Let us denote by $\overline{\theta}(t)$ the solution of (3.10) which corresponds to the solution ${}^T(\dot{z}_0(t), \ddot{z}_0(t))$ of $(3.3)_0$ with $\epsilon = 0$. We show that $\theta_\pm(\pm\epsilon t_0,\epsilon,0)$ and $\overline{\theta}(\pm t_0)$ can be made arbitrarily close by choosing $\epsilon > 0$ small. By definition, $\tan\Theta_-(-\epsilon t_0,\epsilon,0) = -[-f_u(Z(-\epsilon t_0,\epsilon), -\epsilon t_0)]^{1/2}$. On the other hand, $\tan\overline{\theta}(-t_0) = -\ddot{z}_0(-t_0)/\dot{z}_0(-t_0)$. One should notice that both $\tan\Theta_-(-\epsilon t_0,\epsilon,0)$ and $\tan\overline{\theta}(-t_0)$ have one and the same sign (negative in this case). Hence, we estimate

(3.11) $$\begin{aligned} &\tan^2\Theta_-(-\epsilon t_0,\epsilon,0) - \tan^2\overline{\theta}(-t_0) \\ &= -f_u(Z(-\epsilon t_0,\epsilon), -\epsilon t_0) + [f(z_0(-t_0),0)]^2 \Big/ \left[2\int_0^{z_0(-t_0)} f(u,0)du\right] \end{aligned}$$

By employing the expansion $f(u,0) = f_u(0,0)u + O(|u|^2)$ near $u = 0$, the second on the right of (3.11) reduces to $f_u(0,0) + O(|z_0(-t_0)|)$. Hence, we can continue formula (3.11) as:

$$= -\epsilon t_0 f_{ux}(0,x^*) + f_{uu}(u^*,-\epsilon t_0)[-Z(-t_0,\epsilon)]$$

where $x^* \in [-1,1]$ and $u^* \in (0,1)$ are appropriate values. This gives:

$$|\Theta_-(-\epsilon t_0,\epsilon,0) - \overline{\theta}(-t_0)| \leq C[\epsilon t_0 + e^{-2\beta t_0}]$$

for some consant $C > 0$. The same type of arguments gives:

$$|\Theta_+(\epsilon t_0,\epsilon,0) - \overline{\theta}(t_0)| \leq C(\epsilon t_0 + e^{-2\beta t_0}).$$

Combining these estimates with Lemma 3.2 (i), (ii), one obtains

(3.12) $$\begin{cases} |\theta_-(-\epsilon t_0,\epsilon,0) - \overline{\theta}(-t_0)| \leq C(\epsilon t_0 + e^{-2\beta t_0}) + k_3\epsilon + k_4 e^{-2\overline{\beta}(1-\epsilon t_0)/\epsilon} \\ |\theta_+(\epsilon t_0,\epsilon,0) - \overline{\theta}(t_0)| \leq C(\epsilon t_0 + e^{-2\beta t_0}) + k_3\epsilon + k_4 e^{2\overline{\beta}(\epsilon t_0 - 1)/\epsilon}. \end{cases}$$

If $\mathbf{\Theta}_\pm(t,\epsilon,\lambda) \equiv \theta_\pm(\epsilon t,\epsilon,\lambda)$, then $\mathbf{\Theta}_\pm(t,\epsilon,\lambda)$ are solutions of $(3.5\text{-b})_0$ with $\mathbf{\Theta}_\pm(\pm\epsilon^{-1},\epsilon,\lambda) = 0$. Since the right hand side of $(3.5\text{-b})_\lambda$ is bounded and solutions of differential equations depend continuously on initial data and parameters involved, relation (3.12) implies that

$$(3.13)\qquad \begin{cases} |\mathbf{\Theta}_-(0,\epsilon,0) - \bar{\mathbf{\Theta}}(0)| \leq \bar{C}t_0(\epsilon t_0 + e^{-2\beta t_0} + k_3\epsilon + k_4 e^{-2\bar{\beta}(1-\epsilon t_0)/\epsilon}) \\ |\mathbf{\Theta}_+(0,\epsilon,0) - \bar{\mathbf{\Theta}}(0)| \leq \bar{C}t_0(\epsilon t_0 + e^{-2\beta t_0} + k_3\epsilon + k_4 e^{-2\bar{\beta}(1-\epsilon t_0)/\epsilon}). \end{cases}$$

The positive constant $\bar{C}$ does not depend on t_0, ϵ so long as $t_0 > \bar{t}_0$, $0 < \epsilon \leq \bar{\epsilon}_0$. For any $\lambda \in (0,\beta^2/2]$, one has

$$(3.14)_+\qquad \begin{cases} \theta_-(0,\epsilon,0) = \mathbf{\Theta}_-(0,\epsilon,0) > \mathbf{\Theta}_-(0,\epsilon,\lambda) = \theta_-(0,\epsilon,\lambda) \\ \theta_+(0,\epsilon,0) = \mathbf{\Theta}_+(0,\epsilon,0) < \mathbf{\Theta}_+(0,\epsilon,\lambda) \end{cases}$$

and

$$(3.14)_-\qquad \begin{cases} \theta_-(0,\epsilon,0) = \mathbf{\Theta}_-(0,\epsilon,0) < \mathbf{\Theta}_-(0,\epsilon,-\lambda) = \theta_-(0,\epsilon,-\lambda) \\ \theta_+(0,\epsilon,0) = \mathbf{\Theta}_+(0,\epsilon,0) > \mathbf{\Theta}_+(0,\epsilon,-\lambda) = \theta_+(0,\epsilon,-\lambda). \end{cases}$$

Now choose $t_0 = t_0(\epsilon) \to \infty$ as $\epsilon \to 0$ so that $t_0^2(\epsilon)\epsilon \to 0$ as $\epsilon \to 0$. Then relation (3.13) implies that: $|\theta_-(0,\epsilon,0) - \theta_+(0,\epsilon,0)| = |\mathbf{\Theta}_-(0,\epsilon,0) - \mathbf{\Theta}_+(0,\epsilon,0)| \to 0$ as $\epsilon \to 0$. Therefore, $(3.14)_\pm$ imply that, for any $\lambda \in (0,\beta^2/2]$, there exists an $\epsilon_0(\lambda) > 0$ such that

$$(3.15)\qquad \begin{cases} \theta_-(0,\epsilon,\lambda) < \theta_+(0,\epsilon,\lambda) \\ \theta_-(0,\epsilon,-\lambda) > \theta_+(0,\epsilon,-\lambda) \end{cases} \quad \text{for} \quad \epsilon \in (0,\epsilon_0(\lambda)].$$

The inequalities in (3.15) imply that $\lim \lambda_1(\epsilon) = 0$. For, if $\lambda_1(\epsilon) \geq \delta > 0$ for $\epsilon > 0$ small, then $\theta_-(0,\epsilon,\lambda_1(\epsilon)) \leq \theta_-(0,\epsilon,\delta) < \theta_+(0,\epsilon,\delta) \leq \theta_+(0,\epsilon,\lambda_1(\epsilon))$ for $\epsilon \in (0,\epsilon_0(\delta)]$ which is a contradiction, since $\lambda_1(\epsilon)$ being the first eigenvalue implies $\theta_-(0,\epsilon,\lambda_1(\epsilon) = \theta_+(0,\epsilon,\lambda_1(\epsilon))$. This proves part (i) of Theorem 3.1.

Part (ii) of Theorem 3.1 is easily obtained from $(3.4\text{-a})_{\lambda_1(\epsilon)}$ and Lemma 3.2.

In order to prove part (iii), we need the following elementary lemma.

Lemma 3.3. *Let* $\bar{t}_0$, $\bar{\epsilon}_0$ *be fixed so that the condition* (3.6) *is satisfied. Then the following estimate is valid.*

$$0 \leq \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) \leq \mu/(\tan^2\Theta_0 + 1)$$

$$0 \leq \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon)) - \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) \leq \mu/(\tan^2\Theta_0 + 1)$$

for $\epsilon \in (0,\bar{\epsilon}_0]$, $t_0 \geq \bar{t}_0$, *and* $\mu \in [0,\beta^2/4]$.

Proof. Since $-\pi/2 < \Theta_- \leq -\Theta_0 < 0$ and

$$\tan \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) - \tan \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) = \mu,$$

an application of the mean value theorem implies

$$\Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) \leq \mu/(1 + \tan^2\Theta_0).$$

The statement for Θ_+ follows from the same type of arguments.

Now consider the equation

$$\epsilon\theta' = [f_u(U(x,\epsilon),x) - \lambda_1(\epsilon) + \mu] \cos^2\theta + \sin^2\theta \tag{3.16}$$

for $\mu \in [0,\beta^2/4]$. Let us denote by $\theta_\mu(x,\epsilon)$ the solution of (3.16) with $\theta_\mu(-1,\epsilon) = 0$. Hence, $\theta_0(x,\epsilon)$ corresponds to the first eigenvalue of $\mathfrak{L}^\epsilon$. We shall give an lower bound for μ for which $\theta_\mu(1,\epsilon) = \pi$ holds. For sufficiently small $\epsilon > 0$, say, $\epsilon \in (0,\epsilon_0]$, for some $\epsilon_0 > 0$, $|\lambda_1(\epsilon) - \mu| < \beta^2/2$ is satisfied for $\mu \in [0,\beta^2/4]$ and the coefficient of $\cos^2\theta$ in the right side of (3.16) satisfies (3.7). Therefore Lemma 3.2 applies. In order to have $\theta_\mu(1,\epsilon) = \pi$, it is necessary that:

$$\theta_\mu(\epsilon t_0,\epsilon) > \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon) - \mu) - k_3\epsilon + \pi.$$

On the other hand, Lemma 3.2 implies

$$\theta_\mu(-\epsilon t_0,\epsilon) < \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon) - \mu) + 2k_3\epsilon.$$

Therefore, the inequality

$$\theta_\mu(\epsilon t_0,\epsilon) - \theta_\mu(-\epsilon t_0,\epsilon) > \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(t)-\mu) - 3k_3\epsilon + \pi \tag{3.17}$$

must be satisfied. Since $\mu > 0$, $\epsilon\theta_\mu'$ is estimated as

$$\epsilon\theta_\mu' \leq [f_u(U(x,\epsilon),x)-\lambda_1(\epsilon)] \cos^2\theta_\mu + \sin^2\theta + \mu.$$

Comparing the equation with the equation for θ_0;

$$\epsilon\theta_0' = [f_u(U(x,\epsilon),x)-\lambda_1(\epsilon)] \cos^2\theta_0 + \sin^2\theta_0$$

one can easily verify the existence of a constant $\bar{k} > 0$ such that

$$\epsilon(\theta_\mu - \theta_0)' \leq \bar{k}(\theta_\mu - \theta_0) + \mu. \tag{3.18}$$

Solving the differential inequality (3.18) over the interval $[-\epsilon t_0,\epsilon t_0]$, one obtains

$$\begin{aligned}\theta_\mu(\epsilon t_0,\epsilon) &- \theta_\mu(-\epsilon t_0,\epsilon) \\ &< \theta_0(\epsilon t_0,\epsilon) - \theta_0(-\epsilon t_0,\epsilon) + [\theta_\mu(-\epsilon t_0,\epsilon) - \theta_0(-\epsilon t_0,\epsilon) + 2t_0\mu]e^{2\bar{k}t_0}.\end{aligned} \tag{3.19}$$

Applying Lemma 3.2 again, one easily finds that:

$$\theta_\mu(-\epsilon t_0,\epsilon) - \theta_0(-\epsilon t_0,\epsilon) < \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)-\mu) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) + 2k_3\epsilon$$

and further applying Lemma 3.3,

$$\theta_\mu(-\epsilon t_0,\epsilon) - \theta_0(-\epsilon t_0,\epsilon) < \mu/(1 + \tan^2\Theta_0) + 2k_3\epsilon. \tag{3.20}$$

Lemma 3.2 also implies that

$$\theta_0(\epsilon t_0,\epsilon) - \theta_0(-\epsilon t_0,\epsilon) < \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon)) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) + 2k_3\epsilon. \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), one obtains

$$\begin{aligned}\theta_\mu(\epsilon t_0,\epsilon) &- \theta_\mu(-\epsilon t_0,\epsilon) \\ &< \Theta_+(\epsilon t_0,\epsilon,\lambda_1(\epsilon)) - \Theta_-(-\epsilon t_0,\epsilon,\lambda_1(\epsilon)) + 2k_3\epsilon \\ &\qquad + [\mu/(1 + \tan^2\Theta_0) + 2k_3\epsilon + 2t_0\mu]e^{2\bar{k}t_0}.\end{aligned} \tag{3.22}$$

Combining (3.17) and (3.22) and using Lemma 3.3 again one obtains,

(3.23) $$\pi < \Phi(\mu) + 2k_3(e^{2\bar{k}t_0} + 5)\epsilon$$

in which $\Phi(\mu) = [(2 + e^{2\bar{k}t_0})/(1 + \tan^2\Theta_0) + 2t_0e^{2\bar{k}t_0}]\mu$. Hence, for $\epsilon \in (0,\epsilon_0]$, with $\epsilon_0 > 0$ sufficiently small, (3.23) implies $\pi/2 < \Phi(\mu)$. There are two cases:

Case 1) $$\Phi(\beta^2/4) < \pi/2.$$

This means: $\lambda_2(\epsilon) = \lambda_1(\epsilon) - \mu < \lambda_1(\epsilon) - \beta^2/4 < -\beta^2/8$ for $\epsilon \in (0,\epsilon_0]$.

Case 2). There exists a unique $\mu_0 \in (0,\beta^2/8]$ such that

$$\Phi(2\mu_0) = \pi/2.$$

This means $\lambda_2(\epsilon) = \lambda_1(\epsilon) - \mu < \lambda_1(\epsilon) - 2\mu_0 < -\mu_0$ for $\epsilon \in (0,\epsilon_0]$. This completes the proof of part (iii) of theorem 3.1.

We shall refine the estimate on $\lambda_1(\epsilon)$ as $\epsilon \to 0$, in the following

Lemma 3.4. *The first eigenvalue* $\lambda_1(\epsilon)$ *of* $\mathfrak{L}^\epsilon$ *satisfies*

$$\lambda_1(\epsilon) = K_1\epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

where $K_1 = -J'(0)/\int_{-\infty}^{\infty} \dot{z}_0(t)^2dt$ *and* $\lim_{\epsilon\to 0} o(\epsilon)/\epsilon = 0$.

Proof. Let $\phi_1(x,\epsilon)$ be a principal eigenfunction of $\mathfrak{L}^\epsilon$ normalized so that $\phi_1(0,\epsilon) = \dot{z}_0(0)$. This normalization is always possible since $\phi_1(x,\epsilon)$ is of constant sign on $[-1,1]$ and $\dot{z}_0(0) \neq 0$.

If $\Phi_1(t,\epsilon) = \phi_1(\epsilon t,\epsilon)$, then Φ_1 satisfies the equation

(3.24) $$\ddot{\Phi}_1 + f_u(\tilde{U}(t,\epsilon),\epsilon t)\Phi = \lambda_1(\epsilon)\Phi.$$

By Theorem 3.1 (ii) and the normalization above, $|\Phi_1(\cdot,\epsilon)|_0$ is bounded for $\epsilon \in (0,\epsilon_0]$, as well as $|f_u(\tilde{U}(\cdot,\epsilon),\epsilon t)|_0$ and $\lambda_1(\epsilon)$. The equation (3.24) implies that $|\ddot{\Phi}(\cdot,\epsilon)|_0$ is bounded. The interpolation inequality $|u'|_0 \leq \alpha|u|_0 + (2/\alpha)|u''|_0$ for any $\alpha > 0$ implies that $|\dot{\Phi}(\cdot,\epsilon)|_0$ is bounded. Applying

the Ascoli-Arzela's theorem and the equation (3.24) repeatedly, one finds that $(\lambda_1(\epsilon), \Phi(\cdot,\epsilon))$ is precompact in $R \times C^2_{loc}(R)$.

On the other hand, *the proof of Theorem* 3.1 *(i) shows that* $\theta_-(0,\epsilon,\lambda_1(\epsilon)) \to \theta(0)$ as $\epsilon \to 0$ and hence

$$(\Phi(0,\epsilon),\ \dot{\Phi}(0,\epsilon)) \to (\dot{z}_0(0),\ \ddot{z}_0(0)) \quad \text{as} \quad \epsilon \to 0.$$

Since the solution $\Phi(t,\epsilon)$ depends continuously on the initial data $(\Phi(0,\epsilon), \dot{\Phi}(0,\epsilon))$ and the parameter ϵ, the only possible limit of $\Phi(\cdot,\epsilon)$ as $\epsilon \to 0$ in $C^2_{loc}(R)$ is $\dot{z}_0(\cdot)$.

Now multiply (3.24) by $\dot{z}_0(t)$ and integrate over the interval $[-{}^1/4\epsilon, {}^1/4\epsilon]$ by parts to obtain

$$\begin{aligned}
&\lambda_1(\epsilon) \int_{-1/4\epsilon}^{1/4\epsilon} \Phi(t,\epsilon)\dot{z}_0(t)dt \\
&= [\dot{z}_0(t)\dot{\Phi}(t,\epsilon) - \ddot{z}_0(t)\Phi(t)]_{t=-1/4\epsilon}^{t=1/4\epsilon} \\
&\quad + \int_{-1/4\epsilon}^{1/4\epsilon} [\dddot{z}_0 + f_u(Z(t,\epsilon),\epsilon t)\dot{z}_0]\Phi(t,\epsilon)dt
\end{aligned}$$

in which $\tilde{U}(t,\epsilon) = Z(t,\epsilon)$ for $|t| \leq 1/4\epsilon$, is used.

Substituting $\dddot{z}_0 = -f_u(z_0(t), 0)\dot{z}_0(t)$, one obtains

$$\begin{aligned}
&\lambda_1(\epsilon) \int_{-1/4\epsilon}^{1/4\epsilon} \dot{z}_0(t)\Phi(t,\epsilon)dt \\
&\qquad = O(e^{-\beta/\epsilon}) + \int_{-1/4\epsilon}^{1/4\epsilon} [f_u(Z(t,\epsilon),\epsilon t) - f_u(z_0(t),0)]\dot{z}_0(t)\Phi(t,\epsilon)dt.
\end{aligned}$$

The first term on the right side is obtained from the decay estimates in Theorem 3.1 (ii) and in (2.7). By the Lebesgue's dominated convergence theorem

$$\left[\lim_{\epsilon\to 0}\frac{\lambda_1(\epsilon)}{\epsilon}\right]\int_{-\infty}^{\infty}\dot z_0(t)^2dt$$

$$= \int_{-\infty}^{\infty}[f_{uu}(z_0(t),0)z_1(t) + f_{ux}(z_0(t),0)t]\dot z_0(t)^2dt.$$

The last term is simplified by integration by parts:

$$\int_{-\infty}^{\infty}f_{uu}(z_0(t),0)z_1(t)\dot z_0(t)^2dt + \int_{-\infty}^{\infty}f_{ux}(z_0(t),0)t\dot z_0(t)^2dt$$

$$= - \int_{-\infty}^{\infty}f_u(z_0(t),0)[z_1(t)\ddot z_0(t) + \dot z_1(t)\dot z_0(t)]dt$$

$$- \int_{-\infty}^{\infty}f_x(z_0(t),0)\dot z_0(t)dt - \int_{-\infty}^{\infty}f_x(z_0(t),0)t\ddot z_0(t)dt$$

$$= - \int_{-\infty}^{\infty}f_x(z_0(t),0)\dot z_0(t)dt$$

$$- \int_{-\infty}^{\infty}[f_u(z_0(t),0)z_1(t) + f_x(z_0(t),0)t]\ddot z_0(t)dt$$

$$+ \int_{-\infty}^{\infty}\ddot z_1(t)f(z_0(t),0)dt$$

$$= - \int_0^1 f_x(u,0)du + \int_{-\infty}^{\infty}[\ddot z_1(t) + f_u(z_0(t),0)z_1 + f_x(z_0(t),0)t]f(z_0(t),0)dt$$

$$= - J'(0).$$

This completes the proof of Lemma 3.4.

4. The method of Liapunov-Schmidt.

In this section, we use the method of Liapunov-Schmidt to obtain the existence of equilibrium solutions of (2.13); the solutions of

$$\mathfrak{L}^{\epsilon}u + G(\epsilon) + F(u,\epsilon) = 0 \tag{4.1}$$

with $G(\epsilon)$, $F(u,\epsilon)$ defined in (2.15), (2.16).

Let $E : Y \to Y$ be a continuous projection onto the span of $\phi_1(\cdot,\epsilon)$, the principal eigenfunction of $\mathfrak{L}^{\epsilon}$ normalized so that $\phi_1(0,\epsilon) = \dot{z}_0(0)$. E is given explicitly by:

$$Eu = \langle u,\phi_1(\cdot,\epsilon)\rangle \phi_1(\cdot,\epsilon)/\|\phi_1(\cdot,\epsilon)\|^2_{L^2(-1,1)} \tag{4.2}$$

in which $\langle u,\phi_1(\cdot,\epsilon)\rangle = \int_{-1}^{1} u(x)\phi_1(x,\epsilon)dx$. Let Y_1 and X_1 be the null spaces of E in Y and X, respectively. Associated with the projection E, one has the following decompositions

$$Y = [\phi_1(\epsilon)] \oplus Y_1,\ X = [\phi_1(\epsilon)] \oplus X_1. \tag{4.3}$$

One should notice that $Y_1 = \mathcal{R}(\mathfrak{L}^{\epsilon}) = \mathfrak{L}^{\epsilon}X_1$ and that $\mathfrak{L}^{\epsilon} : X_1 \to Y_1$ is a one-to-one mapping. In accordance with the decompositions in (4.3), the problem (4.1) is recast as

$$\begin{cases} \text{(i)} & \mathfrak{L}^{\epsilon}v + (I-E)G(\epsilon) + (I-E)F(\alpha\phi_1(\epsilon) + v,\epsilon) = 0 \\ \text{(ii)} & \lambda_1(\epsilon)\alpha\phi_1(\epsilon) + EG(\epsilon) + EF(\alpha\phi_1(\epsilon) + v,\epsilon) = 0 \end{cases} \tag{4.4}$$

where u was replaced by $u = \alpha\phi_1(x;\epsilon) + v$, with $\alpha \in \mathbb{R}$, $v \in X_1$.

Lemma 4.1. *There exists an* $\epsilon_0 > 0$ *such that, for* $\epsilon \in (0,\epsilon_0]$ *and* $p \in Y_1$, *the equation* $\mathfrak{L}^{\epsilon}v = p$ *has a unique solution* $v = v(p) \in X_1$. *Moreover, there exists a constant* $k > 0$ such that

$$|v(p)|_{2,\epsilon} \leq k|p|_0, \quad \text{for} \quad \epsilon \in (0,\epsilon_0],\ p \in Y_1 \tag{4.5}$$

Proof. Take an $\epsilon_0 > 0$ so that Theorem 3.1 is valid for $\epsilon \in (0,\epsilon_0]$. The first part is a consequence of the fact that $\mathfrak{L}^\epsilon : X_1 \to Y_1$ is bijective. As for the estimate (4.5), Theorem 3.1 (iii) implies that $\|(\mathfrak{L}^\epsilon)^{-1}\|_{B(Y_1,Y_1)} \leq \mu_0^{-1}$ since $\mathfrak{L}^\epsilon$ is self-adjoint; hence, $|v(p)|_0 \leq |p|_0/\mu_0$. Then, the equation $\mathfrak{L}^\epsilon v = p$ implies that $\epsilon^2|v(p)''|_0 \leq C_0|p|_0$ for some positive constant C_0 which is independent of $\epsilon \in (0,\epsilon_0]$. Now, from the interpolation inequality

$$|v'|_0 \leq \epsilon|v''|_0 + 2\epsilon^{-1}|v|_0 \quad \text{for any} \quad \epsilon > 0,$$

one obtains the existence of a constant $k > 0$ for which the estimate (4.5) is true. This completes the proof of the lemma.

Lemma 4.2. *The equation* (4,4) (i) *is uniquely solved in* v *as a function* $v = v^*(\alpha,\epsilon)$ *of* $(\alpha,\epsilon) \in R \times R_+$ *in a neighborhood of* $(\alpha,\epsilon) = (0,0)$, *smooth in* α, *and* $|v^*(\alpha,\epsilon)|_{2,\epsilon} = (\alpha^2 + \epsilon^2)$ *as* $|\alpha| + \epsilon \to 0$.

Proof. The lemma is proved by a standard application of the contraction maping principle.

Let $\mathcal{F} : Y_1 \times \mathbb{R} \times (0,\epsilon_0] \to Y_1$ be defined by

$$\mathcal{F}(v,\alpha,\epsilon) = K^\epsilon(I-E)[G(\epsilon) + F(\alpha\phi_1(\epsilon) + v,\epsilon)]$$

in which $K^\epsilon = -[\mathfrak{L}^\epsilon|Y_1]^{-1}$ and ϵ_0 is sufficiently small to permit the application of Lemma 4.1 and Theorem 3.1. From Lemmas 2.1 and 4.1, we have

$$|(I-E)G(\epsilon)|_0 = O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

From Lemma 4.1 and the definition of $F(u,\epsilon)$ in (2.16), we also have

$$|(I-E)F(\alpha\phi_1(\epsilon) + v,\epsilon)|_0 = O(|\alpha| + |v|_0)^2$$

as $\alpha, |v|_0 \to 0$. Therefore, there are positive constants c_0, k_5 such that, for $0 < \epsilon \leq \epsilon_0$, $|v|_0 \leq c_0$, $|\alpha| \leq c_0$

(4.6) $$|\mathcal{F}(v,\alpha,\epsilon)|_{2,\epsilon} \leq k_5(\epsilon^2 + (|\alpha| + |v|_0)^2)$$

and for $|v_1|_0$, $|v_2|_0 \leq c_0$,

(4.7) $$|\mathcal{F}(v_1,\alpha,\epsilon) - \mathcal{F}(v_2,\alpha,\epsilon)|_{2,\epsilon} \leq k_5(|v_1|_0 + |v_2|_0)|v_1 - v_2|_0$$

If $Y_1(r) \equiv \{v \in Y_1;\ |v|_0 \leq r\}$ for $r > 0$, then, replacing c_0, ϵ_0 by smaller values, if necessary, one can find $r > 0$ so small that the following inequalities are satisfied

(4.8) $$k_5\{\epsilon_0^2 + (c_0 + r)^2\} < r \quad \text{and} \quad 2k_5 r < 1.$$

For such a choice of r as above, and $\alpha \in [-c_0,c_0]$, $\epsilon \in (0,\epsilon_0]$ the mapping

$$\mathcal{F}(\cdot,\alpha,\epsilon) : Y_1(r) \to Y_1(r)$$

is a contraction mapping. Hence, the existence of the function $v^*(\alpha,\epsilon)$, $\alpha \in [-c_0,c_0]$, $\epsilon \in (0,\epsilon_0]$, is ensured. The order estimate on $|v^*(\alpha,\epsilon)|_{2,\epsilon}$, as $|\alpha| + \epsilon \to 0$ is obtained from (4.6). The proof is complete.

We are now in a position to state an existence theorem for equilibrium solutins of (1.1), (1.2) with a single transition layer.

Theorem 4.3. *If* (A-1) - (A-2) *and* (A-3) *are satisfied, then there is a family of equilibrium solutions* $u_+(x,\epsilon)$ *of* (1.1), (1.2) *with the following properties.*

(i) $$\lim_{\epsilon \to 0} u_+(x,\epsilon) = \begin{cases} 0 & \textit{compact uniformly on } [-1,0) \\ 1 & \textit{compact uniformly on } (0,1] \end{cases}$$

and

$$|u_+(\cdot,\epsilon) - U(\cdot,\epsilon)|_{2,\epsilon} = O(\epsilon) \text{ as } \epsilon \to 0$$

(ii) *If* $\mathfrak{L}_+^\epsilon : X \to Y$ *is defined by* $\mathfrak{L}_+^\epsilon v \equiv \epsilon^2 v'' + f_u(u_+(x,\epsilon),x)v$, *then the principal eigenvalue* $\lambda_1^*(\epsilon)$ *of* $\mathfrak{L}_+^\epsilon$ *satisfies*

$$\lambda_1^*(\epsilon) = K_1\epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0$$

and there exists a positive constant μ_0 *such that*

$$\lambda_2^*(\epsilon) \leq -\mu_0 \quad \textit{for} \quad \epsilon \in (0,\epsilon_0]$$

where K_1 *is the constant defined in Lemma* 3.4 *and* $\lambda_2^*(\epsilon)$ *is the second eigenvalue of* $\mathfrak{L}_\epsilon^\xi$.

(iii) *There exists a positive constant* d_0 *independent of* $\epsilon \in (0,\epsilon_0]$ *such that* $u_+(x,\epsilon)$ *is monotone increasing over the interval* $[-\epsilon d_0, \epsilon d_0]$.

Proof. Finding such a family of equilibrium solutons of (1.1), (1.2) is equivalent to finding a family of solutions $(\alpha(\epsilon), v(\epsilon))$ of the problem (4.1). By Lemma 4.2, the later one is reduced to solving:

$$B(\alpha,\epsilon) = 0$$

where $B(\alpha,\epsilon)\phi_1(\epsilon) \equiv \lambda_1(\epsilon)\alpha\phi_1(\epsilon) + E[G(\epsilon) + F(\alpha\phi_1(\epsilon) + v^*(\alpha,\epsilon),\epsilon)]$. From Lemmas 3.4, 4.2, it follows that

$$B(\alpha,\epsilon) = \tau_0\epsilon^2 + o(\epsilon^2) + (K_1\epsilon + o(\epsilon))\alpha + o(1)\alpha^2 + O(\alpha^3) \tag{4.9}$$

in which τ_0 is given by:

$$\tau_0 = \int_{-\infty}^{\infty} \dot z_0(t)[\tfrac{1}{2}f_{uu}(z_0(t),0)z_1(t)^2 + f_{ux}(z_0(t),0)tz_1(t) + \tfrac{1}{2} f_{xx}(z_0(t),0)t^2]dt/\|\dot z_0\|^2_{L^2(-1,1)} \tag{4.10}$$

and $o(\epsilon^2)/\epsilon^2 \to 0$ as $\epsilon \to 0$, $o(1) \to 0$ as $\epsilon \to 0$. By applying the implicit function theorem to:

$$(^1/\epsilon^2)B(\epsilon\tilde\alpha,\epsilon) = 0$$

one can show that there is a unique solution $(\alpha^*(\epsilon),\epsilon)$ of $B(\alpha,\epsilon) = 0$ for $|\alpha| < c_0$, $\epsilon \in (0,\epsilon_0]$, which satisfies

$$\alpha^*(\epsilon) = (-\tau_0/K_1)\epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \tag{4.11}$$

Therefore, $(\alpha^*(\epsilon), v^*(\alpha^*(\epsilon),\epsilon),\epsilon)$ is a unique solution of (4.4) for $|\alpha| \leq c_0$, $\epsilon \in (0,\epsilon_0]$.

Now, our desired solution $u_+(x,\epsilon)$ is given by

(4.12) $$u_+(x,\epsilon) = U(x,\epsilon) + \alpha^*(\epsilon)\phi_1(x,\epsilon) + v^*(\alpha^*(\epsilon),\epsilon).$$

Since it follows that $|\phi_1(\cdot,\epsilon)|_{2,\epsilon} \leq k_2|\dot{z}_0(0)|$, $|\alpha^*(\epsilon)| = O(\epsilon)$, and $|v^*(\alpha^*(\epsilon),\epsilon)|_{2,\epsilon} = O(\epsilon^2)$ from Theorem 3.1, Lemma 4.2, and the estimate given above, one obtains

(4.13) $$|u_+(\cdot,\epsilon) - U(\cdot,\epsilon)|_{2,\epsilon} = O(\epsilon)$$

which, along with the construction of $U(\cdot,\epsilon)$, proves part (i).

In order to prove part (ii), one just follows the same procedure as in the proof of Lemma 3.4 to obtain

$$\left[\lim_{\epsilon\to 0} \lambda_1^*(\epsilon)/\epsilon\right]\int_{-\infty}^{\infty}\dot{z}_0(t)^2dt = \int_{-\infty}^{\infty}[f_{uu}(z_0(t),0)\bar{z}_1(t) + f_{ux}(z_0(t),0)t]\dot{z}_0(t)^2dt$$

in which $\bar{z}_1(t) = (-\tau_0/K_1)\dot{z}_0(t) + z_1(t)$. The subsequenct computations in the proof of Lemma 3.4 is valid as well in the present situation, since $\bar{z}_1$ satisfies the non-homogeneous linear equation

$$\ddot{\bar{z}}_1 + f_u(z_0(t),0)\bar{z}_1 + f_x(z_0(t),0)t = 0.$$

Therefore, one can conclude:

$$\lim_{\epsilon\to 0} \lambda_1^*(\epsilon)/\epsilon = K_1.$$

As for the existence of such a constant $\mu_0 > 0$ as to satisfy: $\lambda_2^*(\epsilon) \leq -\mu_0$ for $\epsilon \in (0,\epsilon_0]$, one can follow the arguments which led us to Theorem 3.1 (iii). This completes the proof of part (ii).

To prove part (iii), let us notice that, for $|x| < 1/4$

$$\epsilon u_+'(x,\epsilon) = \dot{z}_0(x/\epsilon) + \epsilon\dot{z}_1(x/\epsilon) + \alpha^*(\epsilon)\epsilon\phi_1'(x,\epsilon)$$
$$+ \epsilon(dv^*(\alpha^*(\epsilon),\epsilon)(x)/dx).$$

Since $|\alpha^*(\epsilon)\phi_1(\cdot,\epsilon) + v^*(\alpha^*(\epsilon),\epsilon)|_{2,\epsilon} = O(\epsilon)$, and $\dot{z}_0(t) > 0$ for $t \in R$, one can choose $d_0 > 0$ so that

$$\inf\{\dot z_0(t);\ |t| \leq d_0\} > |\epsilon \dot z_1(\cdot) + \epsilon\alpha^*(\epsilon)\phi_1'(\cdot,\epsilon) + \epsilon v^*(\alpha^*(\epsilon),\epsilon)'|_0$$

for $\epsilon \in (0,\epsilon_0]$ (by reducing $\epsilon_0 > 0$, if necessary). For this choice of d_0, $u_+'(x,\epsilon) > 0$ for $|x| \leq \epsilon d_0$, $\epsilon \in (0,\epsilon_0]$. This completes the proof of Theorem 4.3.

Remark 4.4. We could construct another family of equilibrium solutions of (1.1), (1.2) with a single transition layer, which, however, "jumps down" from 1 to 0 as x passes zero from left to right. We state this as

Corollary 4.5. *If* (A-1), (A-2) *and* (A-3) *are satisfsied, then there is a family of equilibrium solutions* $u_-(x,\epsilon)$, *for* $\epsilon \in (0,\epsilon_0]$, of (1.1), (1.2) *with the following properties.*

(i)
$$\lim_{\epsilon\to 0} u_-(x,\epsilon) = \begin{cases} 1 & \textit{compact uniformly on } [-1,0) \\ 0 & \textit{compact uniformly on } (0,1] \end{cases}$$

and

$$|u_-(\cdot,\epsilon) - U_-(\cdot,\epsilon)|_{2,\epsilon} = O(\epsilon) \quad \textit{as} \quad \epsilon \to 0$$

(ii) *If* $\mathfrak{L}_-^\epsilon : X \to Y$ *is defined by* $\mathfrak{L}_-^\epsilon v \equiv \epsilon^2 v'' + f_u(u_-(x,\epsilon),x)v$, *then the first eigenvalue* $\lambda_1^*(\epsilon)$ *of* $\mathfrak{L}_-^\epsilon$ *satisfies*

$$\lambda_1^*(\epsilon) = -K_1\epsilon + 0(\epsilon) \quad \textit{as} \quad \epsilon \to 0$$

and there exists a positive constant μ_0 *such that*

$$\lambda_2^*(\epsilon) \leq -\mu_0 \quad \textit{for} \quad \epsilon \in (0,\epsilon_0]$$

where $\lambda_2^*(\epsilon)$ *is the the second eigenvalue of* $\mathfrak{L}_-^\epsilon$.

(iii) *There exists a positive constant* d_0 *independent of* $\epsilon \in (0,\epsilon_0]$ *such that* $u_-(x,\epsilon)$ *is monotone decreasing over the interval* $[-\epsilon d_0,\epsilon d_0]$.

Proof. First of all, we have to construct our approximate solution $U_-(x,\epsilon)$. This is achieved exactly in the same way as was done for $U(x,\epsilon)$, except for the following two steps:

1) z_0, a solution of (2.3), is chosen such that

$$z_0(-\infty) = 1, \; z_0(+\infty) = 0$$

2) $U_-(x,\epsilon) = \zeta_0(x)Z(x/\epsilon,\epsilon) + \zeta_-(x)$

in which $\zeta_-(x)$ is a smooth cut-off function define by:

$$\zeta_-(x) = \begin{cases} 0 \;, & x \in [0,1] \\ 1-\zeta_0(x), & x \in [-1,0] \end{cases}$$

The remainder of the proof of Corollary 4.5 is identical to that of Theorem 4.3.

Remark 4.6. Theorem 4.3, Corollary 4.5 give not only the existence of equilibrium solutions of (1.1), (1.2), but also their stability properties.

Theorem 4.7. The equilibrium solutions $u_\pm(\cdot,\epsilon)$ are asymptotically stable for $\epsilon > 0$ small if

(4.14) $$u_\pm'(0,\epsilon)J'(0) > 0$$

and unstable if

(4.15) $$u_\pm'(0,\epsilon)J'(0) < 0.$$

Remark 4.7. In the statement of Theorem 4.3 (i), we could slightly improve the modulus of approximation $|u_+(\cdot,\epsilon) - U_+(\cdot,\epsilon)|_{2,\epsilon}$ by choosing $z_1(t)$ carefully. When $z_0(t)$ is chosen so as to satisfy the condition (2.8), the problem (2.4), (2.6) has a one-parameter family of solutions $z_1(t) = c\dot{z}_0(t) + z_1^*(t)$, where

$z_1^*(t)$ is a unique solution of the problem (2.4), (2.6) normalized so that $z_1^*(0) = 0$, $\dot{z}_1^*(0) = 0$ hold. We shall show that, by choosing appropriately the coefficent c of $\dot{z}_0(t)$ in the expression of $z_1(t)$, the number τ_0 in (4.10) can be made equal to zero. This, in turn, implies that $|u_+(\cdot,\epsilon) - U(\cdot,\epsilon)|_{2,\epsilon} = o(\epsilon)$, in view of the proof of Theorem 4.3. Now, $\|\dot{z}_0\|_{L^2}^2 \tau_0$ is written as

$$\|\dot{z}_0\|_{L^2}^2 \tau_0 = c^2 I_2/2 + I_1 c + I_0$$

in which

$$I_2 = \int_{-\infty}^{\infty} f_{uu}(z_0(t),0)\dot{z}_0(t)^3 dt$$

$$I_1 = \int_{-\infty}^{\infty} [f_{uu}(z_0(t),0)\dot{z}_0(t)^2 z_1^*(t) + f_{ux}(z_0(t),0)t\dot{z}_0(t)^2]dt$$

and I_0 is a constant which does not depend on c. Integrating by parts, one easily obtains

$$I_2 = \int_0^1 f_{uu}(u,0)(-2F(u))du = -2\int_0^1 f_u(u,0)f(u,0)du = 0$$

and

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} f(z_0(t),0)[\ddot{z}_1^*(t) + f_u(z_0(t),0)z_1^*(t) + f_x(z_0(t),0)t]dt \\ &\quad - J'(0) \\ &= -J'(0) \neq 0. \end{aligned}$$

Therefore, by setting $c = I_0/J'(0)$, one can make $\tau_0 = 0$.

5. Several transition layers

In this section, we extend our previous results to the cases in which several transition layers can occur. We consider the problem (1.1), (1.2) under the assumptions (A-1), (A-2), and (A-3)$'$:

(A-3)$'$. *If* $J(x) = \int_0^1 f(u,x)du$, $x \in [-1,1]$ *as before, then there exist* n *points* $x_i \in (-1,1)$, $x_i < x_{i+1}$, $i = 0,1,\cdots,n$ *with* $x_0 = -1$, $x_{n+1} = 1$, *such that the following conditions are satisfied:*

(i) $J(x_i) = 0 \quad i = 1,\cdots,n$

(ii) $dJ(x)/dx\,|_{x=x_i} \neq 0$, $i = 1,\cdots,n$

(iii) $\int_0^u f(s,x_i)ds < 0$ *for* $u \in (0,1)$, $i = 1,\cdots,n$.

We intend to construct a family of equilibrium solutions of (1.1), (1.2) which exhibits transition layer phenomena at the points $x = x_i$, $i = 1,\cdots,n$.

Let us begin with constructing approximate solutions. In virtue of the results in §2, one can find the solutions of the following equations.

(5.1) $$\ddot{z}^i_{0,\pm} + f(z^i_{0,\pm},x_i) = 0$$

(5.2) $$\ddot{z}^i_{1,\pm} + f_u(z^i_{0,\pm},x_i) + f_x(z^i_{0,\pm},x_i)t = 0$$

with the boundary conditions

(5.3) $$\begin{cases} \lim\limits_{t\to\infty} z^i_{0,+}((-1)^i t) = 0, \ \lim\limits_{t\to\infty} z^i_{0,+}((-1)^{i+1}t) = 1 \\ \lim\limits_{t\to\infty} z^i_{0,-}((-1)^{i-1}t) = 0, \lim\limits_{t\to\infty} z^i_{0,-}((-1)^i t) = 1 \end{cases}$$

(5.4) $z^i_{1,\pm}$ are bounded on $\mathbb{R}$

for each $i \in \{1,\cdots,n\}$. At this stage, $z^i_{1,\pm}$ is not uniquely determined. We impose the additional condition

$$(5.5)\qquad \int_{-\infty}^{\infty} \dot{z}^i_{0,\pm}[(½)f_{uu}(z^i_{0,\pm}(t),x_i)(z^i_{1,\pm}(t))^2 + f_{xu}(z^i_{0,\pm}(t),x_i)tz^i_{1,\pm}(t) + \tfrac{1}{2} f_{xx}(z^i_{0,\pm}(t),x_i)t^2]dt = 0$$

which determines $z^i_{1,\pm}$ uniquely for $i = 1,\cdots,n$. The significance of the condtion (5.5) is clarified in Remark 4.7. When (5.5) is satisfied, the order of the approximate solution is improved. However, even if (5.5) is not satisfied, the stability analysis below is unchanged.

Let d_0 and $\bar{x}_i$, $i = 0,\cdots,n$ be defined by

$$2d_0 = \min\{x_{i+1} - x_i \mid i = 0,\cdots,n\}$$

$$\bar{x}_i = \tfrac{1}{2}(x_{i+1} + x_i), \ i = 0,1,\cdots,n.$$

We also let $Z^i_\pm(t,\epsilon)$ be defined by

$$(5.6)\qquad Z^i_\pm(t,\epsilon) = z^i_{0,\pm}(t) + \epsilon z^i_{1,\pm}(t), \ i = 1,\cdots,n.$$

By using the notations defined above and the smooth cutoff functions ζ_0,ζ_+, and ζ_-, our approximations $U_{n,\pm}(x,\epsilon)$ are defined by:

$$(5.7)\qquad U_{n,\pm}(x,\epsilon) \begin{cases} = 0 \ \ (=1 \text{ for } U_{n,-}) & \text{for } x \in [-1,\bar{x}_1] \\ = \zeta_0((x-x_i)/d_0)Z^i_\pm((x-x_i)/\epsilon,\epsilon) & \text{for } x \in [\bar{x}_i,\bar{x}_{i+1}] \\ \qquad + \zeta_\pm((x-x_i)/d_0) & i = 2j+1, \ j = 0,\cdots,[\frac{n-1}{2}] \\ = \zeta_0((x-x_i)/d_0)Z^i_\pm((x-x_i)/\epsilon,\epsilon) & \text{for } x \in [\bar{x}_i,\bar{x}_{i+1}] \\ \qquad + \zeta_\mp((x-x_i)/d_0) & i = 2j, \ j = 1,\cdots,[^n/2] \\ = 0 \ \ (1 \text{ for } U_{n,-}) & \text{for } x \in [\bar{x}_{n+1},1] \text{ if } n \text{ is even} \\ = 1 \ \ (0 \text{ for } U_{n,-}) & \text{for } x \in [\bar{x}_{n+1},1] \text{ if } n \text{ is odd} \end{cases}$$

The functions $U_{n,\pm}(x,\epsilon)$ will be our approximation to equilibrium solutions

of (1.1), (1.2). Changing variables in (1.1) by: $u \longrightarrow u + U_{n,\pm}$ the new function u is subject to the equation

(5.8) $$\partial u/\partial t = \mathfrak{L}^{\epsilon,n,\pm} u + G^n_{\pm}(\epsilon) + F^n_{\pm}(u,\epsilon)$$

and the boundary conditions in (1.2), where

(5.9) $$\mathfrak{L}^{\epsilon,n,\pm}u = \epsilon^2 u'' + f_u(U_{n,\pm}(x,\epsilon),x)u$$

(5.10) $$G^n_{\pm}(\epsilon)(x) = \epsilon^2 U''_{n,\pm}(x,\epsilon) + f(U_{n,\pm}(x,\epsilon),x)$$

(5.11) $$F^n_{\pm}(u,\epsilon)(x) = f(U_{n,\pm}(x,\epsilon) + u,x) - f(U_{n,\pm}(x,\epsilon),x) - f_u(U_{n,\pm}(x,\epsilon),x)u$$

Since the main line of argument in the sequel is irrelevant to whether we choose $U_{n,+}$ or $U_{n,-}$ as our approximation, we simply denote $U_{n,\pm}$, $\mathfrak{L}^{\epsilon,n,\pm}$, $G^n_{\pm}$ and $F^n_{\pm}$ by U_n, $\mathfrak{L}^{\epsilon,n}$, G^n and F^n respectively. However, the stability property of the equilibirum solutions of (1.1), (1.2) depends on the choice between $U_{n,+}$ and $U_{n,-}$ (see Lemma 5.2, Theorem 5.4).

We first examine some spectral properties of the linear operator $\mathfrak{L}^{\epsilon,n}$.

Lemma 5.1.

(i) *The first* n *eigenvalues of* $\mathfrak{L}^{\epsilon,n}$, $\lambda_1(\epsilon) > \lambda_2(\epsilon) > \cdots > \lambda_n(\epsilon)$, *approach zero as* ϵ *tends to* 0.

(ii) *Let* $\phi_j(x,\epsilon)$, $j = 1,\cdots,n$ *be an eigenfunction of* $\mathfrak{L}^{\epsilon,n}$ *coresponding to* $\lambda_j(\epsilon)$, *then one has:*

$$\phi_j(x_i+\epsilon t,\epsilon)/\phi_j(x_i,\epsilon) \longrightarrow \dot z^i_{0,\pm}(t)/\dot z^i_{0,\pm}(0) \quad as \quad \epsilon \to 0$$

in $C^2_{loc}(\mathbb{R})$, *for* $i,j=1,\cdots,n$.

(iii) *There exists a positive constant* k_0 *such that*

$$|\phi_j(x_i+\epsilon t,\epsilon)| \leq k_0|\phi_j(x_i,\epsilon)|e^{-2\beta|t|} \quad \text{for} \quad |t| \leq d_0/\epsilon$$

(iv) *The remaining eigenvalues of* $\mathfrak{L}^{\epsilon,n}$ *are bounded away from zero; namely, there exists a positive constant* $\mu_0 > 0$ *such that*

$$\lambda_{n+1}(\epsilon) \leq -\mu_0 \quad \textit{for} \quad \epsilon \in (0,\epsilon_0].$$

Proof. The proof is an obvious modificaiton of that of Theorem 3.1.

(i) Let $\theta_-(x,\epsilon,\lambda)$ be the solutions of $(3.4\text{-b})_\lambda$, in which U is replaced by U_n, with initial data $\theta_-(-1,\epsilon,\lambda) = 0$, for $\lambda \in [-\beta^2/2,\beta^2/2]$. It is sufficent to show that, for any $\lambda \in (0,\beta^2/2]$, there exists $\epsilon_0(\lambda) > 0$ such that

$$\theta_-(1,\epsilon,\lambda) < 0, \ \theta_-(1,\epsilon,-\lambda) > (n-1)\pi \tag{5.12}$$

for $\epsilon \in (0,\epsilon_0(\lambda)]$. We shall prove (5.12) for the case of $n = 2$. For the case in which $n \geq 3$, (5.12) follows from repreated application of the arguments below. If $\theta_0(x,\epsilon,\lambda)$ is the solution of $(3.4\text{-b})_\lambda$ with initial data $\theta_0(\bar{x}_2,\epsilon,\lambda) = 0$, then, the proof of Theorem 3.1 (i) implies that $\theta_-(\bar{x}_2,\epsilon,\lambda) < \theta_0(\bar{x}_2,\epsilon,\lambda) = 0$ for $\epsilon \in (0,\bar{\epsilon}_0(\lambda)]$. If $\theta_+(x,\epsilon,\lambda)$ is the solution of $(3.4\text{-b})_\lambda$ with $\theta_+(1,\epsilon,\lambda) = 0$, then, applying the proof of Theorem 3.1 (i) to θ_0 and θ_+, one obtains:

$$\theta_0(1,\epsilon,\lambda) < \theta_+(1,\epsilon,\lambda) = 0 \quad \text{for} \quad \epsilon \in (0,\bar{\bar{\epsilon}}_0(\lambda)].$$

Therefore, for $\epsilon \in (0,\epsilon_0(\lambda)]$, with $\epsilon_0(\lambda) = \min\{\bar{\epsilon}_0(\lambda), \bar{\bar{\epsilon}}_0(\lambda)\}$,

$$\theta_-(1,\epsilon,\lambda) < \theta_0(1,\epsilon,\lambda) < \theta_+(1,\epsilon,\lambda) = 0.$$

On the other hand, if $\theta_0(x,\epsilon,-\lambda)$ is the solution of $(3.4\text{-b})_{-\lambda}$ with $\theta_0(\bar{x}_2,\epsilon,-\lambda) = \pi/2$, then it follows from Theorem 3.1 that $\pi/2 = \theta_0(\bar{x}_2,\epsilon,-\lambda) < \theta_-(\bar{x}_2,\epsilon,-\lambda) < \pi$ for $\epsilon \in (0,\bar{\epsilon}_0(\lambda)]$. Denoting by $\theta_+(x,\epsilon,-\lambda)$ the solution of $(3.4\text{-b})_{-\lambda}$ with $\theta_+(1,\epsilon,-\lambda) = \pi$, and applying the arguments of Theorem 3.1 (i), it follows that $\theta_0(1,\epsilon,-\lambda) > \theta_+(1,\epsilon,-\lambda) = \pi$, for $\epsilon \in$

$(0,\bar{\bar{\epsilon}}_0(\lambda)]$. Therefore, for $\epsilon \in (0,\epsilon_0(\lambda)]$, one obtains

$$\theta_-(1,\epsilon,-\lambda) > \theta_0(1,\epsilon,-\lambda) > \pi$$

where $\epsilon_0(\lambda) = \min\{\bar{\epsilon}_0(\lambda), \bar{\bar{\epsilon}}_0(\lambda)\}$. This completes the proof of part (i)

In view of part (i), part (ii) follows from the proof of Lemma 3.4, and part (iii) follows from the same type of arguments as the proof of Theorem 3.1 (ii).

The proof of part (iv) is essentially the same as that of Theorem 3.1 (iii). In fact, if one defines $\Theta_\pm(x,\epsilon,\lambda)$ for $x \in (\bigcup_{i=1}^{n-1}[x_i + \epsilon t_0, x_{i+1}-\epsilon t_0]) \cup [-1,x_1-\epsilon t_0] \cup [x_n+\epsilon t_0,1]$ by:

$$[f_u(U^n(x,\epsilon),x) - \lambda]\cos^2\Theta_\pm + \sin^2\Theta_\pm = 0$$

$$-\pi/2 < \Theta_-(x,\epsilon,\lambda) \leq -\Theta_0$$

$$\Theta_0 \leq \Theta_+(x,\epsilon,\lambda) < \pi/2$$

$$\text{for} \quad |\lambda| \leq \beta^2/2,\ 0 < \epsilon \leq \bar{\epsilon}_0$$

in which, t_0, $\bar{\epsilon}_0$ are chosen in such a way that the condition (3.7) is satisfied. In order for the value $\lambda_n(\epsilon) - \mu$ to be the $(n+1)^{\text{-th}}$ eigenvalue of $\mathfrak{L}^{\epsilon,n}$, it is necessary that there exists at least one index $j \in \{1,\cdots,n\}$ for which

$$\theta_-(x_j + \epsilon t_0,\epsilon,\lambda_n(\epsilon) - \mu) > \Theta_+(x_j + \epsilon t_0,\epsilon,\lambda_n(\epsilon) - \mu) - k_3\epsilon + \pi.$$

and

$$\theta_-(x_j - \epsilon t_0,\epsilon,\lambda_n(\epsilon) - \mu) < \Theta_-(x_j - \epsilon t_0,\epsilon,\lambda_n(\epsilon) - \mu) + 2k_3\epsilon$$

hold. Now, following the arguments in the proof of Theorem 3.1 (iii), one can find a positive constant μ_0^j such that

$$\lambda_{n+1}(\epsilon) \leq -\mu_0^j \quad \text{for} \quad \epsilon \in (0,\epsilon_0].$$

If we define $\mu_0 = \min(\mu_0^j;\ j = 1,\cdots,n)$, then it follows that $\lambda_{n+1}(\epsilon) \leq -\mu_0$ for $\epsilon \in (0,\epsilon_0]$. This completes the proof of Lemma 5.1.

In the next lemma, the asymptotic estimates on $\lambda_j(\epsilon), j=1,\cdots,n$ are refined. Let us define $\tilde{K}_i^\pm$ by:

$$(5.13)_+ \qquad \tilde{K}_i^+ = (-1)^i \int_0^1 f_x(u,x_i)du \Big/ \int_{-\infty}^{\infty} [\dot{z}_0^i(t)]^2 dt, \ i = 1,\cdots,n$$

and

$$(5.13)_- \qquad \tilde{K}_i^- = (-1)^{i+1} \int_0^1 f_x(u,x_i)du \Big/ \int_{-\infty}^{\infty} [\dot{z}_0^i(t)]^2 dt, \ i = 1,\cdots,n.$$

Lemma 5.2. *Let* $\{K_j^\pm\}_{j=1}^n$ *be a rearrangement of the sequence* $\{\tilde{K}_i^\pm\}_{i=1}^n$ *in such a way as:*

$$K_1^\pm \geqslant K_2^\pm \geqslant \cdots \geqslant K_n^\pm.$$

Then, the first n *eigenvalues* $\lambda_j^\pm(\epsilon)$, $j = 1,\cdots,n$, *of* $\mathfrak{L}^{\epsilon,n,\pm}$ *satisfy:*

$$(5.14)_\pm \qquad \lambda_j^\pm(\epsilon) = \epsilon K_j^\pm + o(\epsilon) \quad as \quad \epsilon \to 0$$

In order to prove this result, we need another lemma. Let $\bar{\Phi}_j(\epsilon), j=1,\cdots,n$ be the first n eigenfunctions of the linear operator $\mathfrak{L}^{\epsilon,n}$ normalized so that

$$(5.15.) \qquad \langle \bar{\Phi}_i(\epsilon), \bar{\Phi}_j(\epsilon) \rangle = \delta_{ij}, \ i,j=1,\cdots,n.$$

where $\langle u,v \rangle = \int_{-1}^1 u(x)v(x)dx$ is the usual scalar product. Because of the normalization (5.15), $\sqrt{\epsilon}\,|\bar{\Phi}_j(\cdot,\epsilon)|_0$, $j = 1,\cdots,n$, are bounded; hence, by reasoning as before, Lemma 5.1 (ii) implies:

$$(5.16) \qquad \sqrt{\epsilon}\ \bar{\Phi}_j(x_i + \epsilon t, \epsilon) \to a_{ji}\dot{z}^i_{0,\pm}(t) \quad \text{in} \quad C^2_{loc}(\mathbb{R}) \quad \text{as} \quad \epsilon \to 0$$

for $j,i=1,\cdots,n$, where a_{ji}, $i,j=1,\cdots,n$, are some real constants.

Lemma 5.3. *If* $A := (a_{ij})_{i,j=1}^n$ *is the* $n \times n$*-matrix with its entries* a_{ij} *defined in* (5.16), *then it follows that:*

$$\det A \neq 0$$

and, in particular, A *is invertible.*

Proof. In virtue of (5.16) and Lemma 5.1 (iii),

$$<\overline{\Phi}_i(\epsilon),\overline{\Phi}_j(\epsilon)> = \sum_{k=1}^{n}\int_{x_k-d_0}^{x_k+d_0} \overline{\Phi}_i(x,\epsilon)\overline{\Phi}_j(x,\epsilon)dx + O(e^{-\beta/\epsilon})$$

$$= \sum_{k=1}^{n} \epsilon\int_{-d_0/\epsilon}^{d_0/\epsilon} \overline{\Phi}_i(x_k + \epsilon t,\epsilon)\overline{\Phi}_j(x_k + \epsilon t,\epsilon)dt + O(e^{-\beta}/\epsilon)$$

$$= \sum_{k=1}^{n} \int_{-d_0/\epsilon}^{d_0/\epsilon} [\sqrt{\epsilon}\,\overline{\Phi}_i(x_k + \epsilon t,\epsilon)][\sqrt{\epsilon}\,\overline{\Phi}_j(x_k + \epsilon t,\epsilon)]dt + O(e^{-\beta}/\epsilon)$$

$$\to \sum_{k=1}^{n} a_{ik}a_{jk}\int_{-\infty}^{\infty} [\dot{z}_0^k(t)]^2dt = \sum_{k=1}^{n} (a_{ik}\|\dot{z}_0^k\|)(a_{jk}\|\dot{z}_0^k\|)$$

as $\epsilon \to 0$ for $i,j = 1,\cdots,n$. In view of the normalization (5.15), one obtains:

$$\sum_{k=1}^{n} \tilde{a}_{ik}\tilde{a}_{jk} = \delta_{ij} \qquad i,j=1,\cdots,n \tag{5.17}$$

where $\tilde{a}_{ik} = a_{ik}\|\dot{z}_{0,\pm}^k\|$. The relations in (5.17) mean that the matrix $\tilde{A} = (\tilde{a}_{ij})$ is an orthogonal matrix, and hence $\det \tilde{A} = 1$. From this, it follows that

$$\det A = (\det \tilde{A}) \prod_{k=1}^{n} \|\dot{z}_0^k\|^{-1} = \prod_{k=1}^{n} \|\dot{z}_0^k\|^{-1} \neq 0.$$

Proof of Lemma 5.2. Consider the eigenvalue problem

$$\mathfrak{L}^{\epsilon,n}u = \lambda u. \tag{5.18}$$

The method of Liapunov-Schmidt will be used in order to obtain a system of n equations which the first n eigenvalues $\lambda_j(\epsilon)$, $j = 1,\cdots,n$, should satisfy. This system of n equations then will be "diagonalized" by using the matrix A which is defined above.

If we let $\psi_i(x,\epsilon) = \epsilon^{-1/2}\zeta_0((x-x_i)/d_0)\dot{z}_{0,\pm}^i((x-x_i)/\epsilon)$ $i=1,\cdots,n$, then $\operatorname{Supp}\psi_i(\epsilon) \cap \operatorname{Supp}\psi_j(\epsilon)$ is empty for $i \neq j$, and therefore, $\{\psi_1(\epsilon),\cdots,\psi_n(\epsilon)\}$ spans an n-dimensional subspace in X (and hence in Y). We define a continuous

projection E_n onto $[\bar{\Phi}_1(\epsilon),\cdots,\bar{\Phi}_n(\epsilon)]$ by

$$(5.19) \qquad E_n u = \sum_{k=1}^{n} \langle\bar{\Phi}_k(\epsilon),u\rangle\bar{\Phi}_k(\epsilon) \quad \text{for} \quad u \in Y.$$

The spaces X and Y are decomposed in accordance with E_n:

$$(5.20) \qquad \begin{cases} X = [\bar{\Phi}_1(\epsilon),\cdots,\bar{\Phi}_n(\epsilon)]\oplus X_n \\ Y = [\bar{\Phi}_1(\epsilon),\cdots,\bar{\Phi}_n(\epsilon)]\oplus Y_n \end{cases}$$

in which X_n and Y_n are the null spaces of E_n in X and Y, respectively. These spaces are also decomposed as:

$$(5.21) \qquad \begin{cases} X = [\psi_1(\epsilon),\cdots,\psi_n(\epsilon)]\oplus X_n \\ Y = [\psi_1(\epsilon),\cdots,\psi_n(\epsilon)]\oplus Y_n \end{cases} \quad \text{for} \quad \epsilon \in (0,\epsilon_0].$$

This is easily seen by observing that:

$$\left[{}^1/\|\dot{z}_0^j\|^2\right] \langle \bar{\Phi}_i(\epsilon), \psi_j(\epsilon)\rangle \to a_{ij} \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad i,j=1,\cdots,n$$

and $A = (a_{ij})$ is invertible. Let $\tilde{E}_n$ be the projection onto $[\psi_1(\epsilon),\cdots,\psi_n(\epsilon)]$ along the subspace Y_n, which is defined by:

$$(5.22) \qquad \tilde{E}_n u = \sum_{k=1}^{n} c_k(u)\psi_k(\epsilon) \quad \text{for} \quad u \in Y$$

in which $c_k(u)$ is determined by:

$$(5.23) \qquad \sum_{k=1}^{n} \langle\bar{\Phi}_i(\epsilon), \psi_k(\epsilon)\rangle\, c_k(u) = \langle\bar{\Phi}_i(\epsilon),u\rangle,\ i=1,\cdots,n\ .$$

If we let $u = \sum_{k=1}^{n} \alpha_k\psi_k + v$ with $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{R}^n$ and $v \in X_n$, then, in terms of decompositions in (5.21), the equation (5.18) is equivalent to:

$$(5.24) \qquad \begin{cases} \text{a)} \quad A(\epsilon,\lambda)\alpha = 0 \\ \text{b)} \quad [\mathfrak{L}^{\epsilon,n}|_{X_n} - \lambda I_{X_n}]v = \sum_{k=1}^{n} \alpha_k Q_k(\epsilon,\lambda) \end{cases}$$

in which $A(\epsilon,\lambda)$ is the n×n-matrix given by:

$$A(\epsilon,\lambda)_{ij} = c_i(\mathfrak{L}^{\epsilon,n}\psi_j(\epsilon) - \lambda\psi_j(\epsilon)),\ i,j=1,\cdots,n$$

and $Q_k(\epsilon,\lambda)$'s are given by:

$$Q_k(\epsilon,\lambda) = (I - \tilde{E}_n)(\mathfrak{L}^{\epsilon,n}\psi_k(\epsilon) - \lambda\psi_k(\epsilon)).$$

We can assume that $|\lambda_j(\epsilon)| < \mu_0/2$ for $\epsilon \in (0,\epsilon_0]$, $j=1,\cdots,n$ where μ_0 and ϵ_0 are the constants which appeared in Lemma 5.1. Then, $[\mathfrak{L}^{\epsilon,n}|_{X_n} - \lambda I_{X_n}]$ is invertible uniformly with respect to $\epsilon \in (0,\epsilon_0]$ and $\lambda \in [-\mu_0/2, \mu_0/2]$, and therefore, (5.24) implies that the first eigenvalues $\lambda_j(\epsilon)$, $j=1,\cdots,n$ of $\mathfrak{L}^{\epsilon,n}$ are the roots of the equation:

$$(5.25) \qquad \det A(\epsilon,\lambda) = 0.$$

Let us now compute the matrix $A(\epsilon,\lambda)$ explicitly. By definition, $A(\epsilon,\lambda)_{ij} = c_i(\mathfrak{L}^{\epsilon,n}\psi_j(\epsilon) - \lambda\psi_j(\epsilon))$ and hence we shall compute both sides of (5.23) with u replaced by $\mathfrak{L}^{\epsilon,n}\psi_j(\epsilon) - \lambda\psi_j(\epsilon)$, $j = 1,\cdots,n$. In order to avoid complicated notation, we simply write z_0^i etc., instead of $z_{0,\pm}^i$, etc.

$$(5.26) \qquad \langle\bar{\Phi}_i, \mathfrak{L}^{\epsilon,n}\ \psi_j - \lambda\psi_j\rangle = \int_{-1}^{1} \bar{\Phi}_i(x,\epsilon)[\mathfrak{L}^{\epsilon,n}\psi_j(x,\epsilon) - \lambda\psi_j(x,\epsilon)]dx$$

$$= \int_{x_j-d_0}^{x_j+d_0} \bar{\Phi}_i(x,\epsilon)[\mathfrak{L}^{\epsilon,n}\psi_j(x,\epsilon) - \lambda\psi_j(x,\epsilon)]dx$$

$$= \int_{-d_0/4\epsilon}^{d_0/4\epsilon} \sqrt{\epsilon}\ \bar{\Phi}_i(x_j + \epsilon t,\epsilon)[\ \dddot{z}_0^j(t) + f_u(Z^j(t,\epsilon),x_j + \epsilon t)\dot{z}_0^j(t) - \lambda\dot{z}_0^j(t)]dt$$

$$+ O(e^{-\beta/2\epsilon})$$

$$(5.27)\qquad \sum_{k=1}^{n} \langle\bar{\Phi}_i,\psi_k\rangle\, c_k(\mathfrak{L}^{\epsilon,n}\psi_j - \lambda\psi_j)$$

$$= \sum_{k=1}^{n} \left\{ \int_{-d_0/4\epsilon}^{d_0/4\epsilon} \sqrt{\epsilon}\, \bar{\Phi}_i(x_k + \epsilon t,\epsilon)\dot{z}_0^k(t)dt + O(e^{-\beta/2\epsilon})\right\} c_k(\mathfrak{L}^{\epsilon,n}\psi_j - \lambda\psi_j)$$

$$= \sum_{k=1}^{n} (a_{ik} + o(1))c_k(\mathfrak{L}^{\epsilon,n}\psi_j(\epsilon) - \lambda\psi_j(\epsilon)) = \sum_{k=1}^{n} (a_{ik} + o(1))A(\epsilon,\lambda)_{kj}.$$

One should notice that the last term in (5.26) is written as:

$$\epsilon\left\{\frac{1}{\epsilon}\int_{-d_0/4\epsilon}^{d_0/4\epsilon} \sqrt{\epsilon}\,\bar{\Phi}_i(x_j + \epsilon t,\epsilon)[f_u(Z^j(t,\epsilon),x_j + \epsilon t) - f_u(z_0^j(t),x_j)-\lambda]\dot{z}_0^j(t)dt + O(e^{-\beta/2\epsilon})\right\}$$

$$= \epsilon a_{ij}\left\{ \int_{-\infty}^{\infty}[f_{uu}(z_0^j(t),x_j)z_i^j(t) + f_{ux}(z_0^j(t),x_j)t](\dot{z}_0^j(t))^2dt - \frac{\lambda}{\epsilon}\int_{-\infty}^{\infty}[\dot{z}_0^j(t)]^2dt + o(1)\right\}.$$

Equating this expression to the last term in (5.27) and using the fact that $A = (a_{ij})$ is invertible, one obtains:

$$(5.28)\qquad \frac{1}{\epsilon}A(\epsilon,\lambda) = \operatorname{diag}(\Phi_1(\epsilon,\lambda),\cdots,\Phi_n(\epsilon,\lambda)) + o(1) \quad \text{as} \quad \epsilon \to 0$$

in which

$$(5.29)\qquad \Phi_j(\epsilon,\lambda) = \int_{-\infty}^{\infty}[f_{uu}(z_{0,\pm}^j(t),x_j)x_j)z_{1,\pm}^j(t) + f_{ux}(z_{0,\pm}^j(t),x_j)t][\dot{z}_{0,\pm}^j(t)]^2dt - (\lambda/\epsilon)\int_{-\infty}^{\infty}[\dot{z}_{0,\pm}^j(t)]^2dt$$

$$= (-1)^{j^*+1}\int_0^1 f_x(s,x)ds - (\lambda/\epsilon)\int_{-\infty}^{\infty}[\dot{z}_{0,\pm}^j(t)]^2dt$$

and $j^* = j - 1$ if $* =$ "+", $j^* = j$ if $* =$ "-". This formula is a consequence of the computations in the proof of Lemma 3.4. Since the first n eignevalues $\lambda_j^*(\epsilon)$, $j = 1,\cdots,n$, are characterized by: $\det A(\epsilon,\lambda_j^*(\epsilon)) = 0$, (5.28) (5.29) imply

that there is a one-to-one mapping

$$p : \{1,\cdots,n\} \to \{1,\cdots,n\}$$

such that:

$$\lim_{\epsilon\to 0} \lambda_j^{\pm}(\epsilon)/\epsilon = (-1)^{p(j)^*+1}\int_0^1 f_x(u,x_{p(j)})du \Big/ \int_{-\infty}^{\infty} [\dot{z}_{0,\pm}^{p(j)}]^2 dt.$$

From the definition of i^* and (5.13), it follows that

$$\lim_{\epsilon\to 0} \lambda_j^{\pm}(\epsilon)/\epsilon = \tilde{K}_{p(j)}^{\pm}, \quad j = 1,\cdots,n.$$

In view of the fact, $\lambda_1^{\pm}(\epsilon) > \lambda_2^{\pm}(\epsilon) > \cdot > \lambda_n^{\pm}(\epsilon)$, this implies:

$$\lim_{\epsilon\to 0} \lambda_j^{\pm}(\epsilon)/\epsilon = K_j^{\pm}, \quad j = 1,\cdots,n$$

completing the proof of Lemma 5.2.

Before stating the main result of this section, we define two subsets Ω_0 and Ω_1 of $[-1,1]$ by:

$$\begin{aligned}
\Omega_0 &= [-1,x_1) \cup (x_2,x_3) \cup \cdots \cup (x_n,1] && \text{if } n \text{ is even} \\
&= [-1,x_1) \cup \cdots \cup (x_{n-1},x_n) && \text{if } n \text{ is odd} \\
\Omega_1 &= (x_1,x_2) \cup \cdots \cup (x_{n-1},x_n) && \text{if } n \text{ is even} \\
&= (x_1,x_2) \cup \cdots \cup (x_{n-2},x_{n-1}) \cup (x_n,1] && \text{if } n \text{ is odd}
\end{aligned}$$

Theorem 5.4. *Suppose the conditions* (A-1), (A-2), *and* (A-3)' *are satisfied. Then there exist a constant* $\epsilon_0 > 0$ *and two families of equilibrium solutions* $u_{n,\pm}(x,\epsilon)$ *of* (1.1), (1.2) *for* $\epsilon \in (0,\epsilon_0]$ *with the following properties:*

(i) $$\lim_{\epsilon\to 0} u_{n,+}(x,\epsilon) = \begin{Bmatrix} 0 & on & \Omega_0 \\ 1 & on & \Omega_1 \end{Bmatrix} \textit{ compact uniformly}$$

$$\lim_{\epsilon\to 0} u_{n,-}(x,\epsilon) = \begin{Bmatrix} 1 & on & \Omega_0 \\ 0 & on & \Omega_1 \end{Bmatrix} \text{ compact uniformly}$$

and $|u_{n,\pm}(\cdot,\epsilon) - U_{n,\pm}(\cdot,\epsilon)|_{2,\epsilon} = o(\epsilon)$ *as* $\epsilon \to 0.$

(ii) *If* $\mathfrak{L}_*^{\epsilon,n,\pm} : X \to Y$ *are defined by:*

$$\mathfrak{L}_*^{\epsilon,n,\pm} v \equiv \epsilon^2 v'' + f_u(u_{n,\pm}(x,\epsilon),x)v$$

then, the first eigenvalues $\lambda_j(\epsilon,n,\pm)$, $j = 1,\cdots,n$, of $\mathfrak{L}_*^{\epsilon,n,\pm}$ *satisfy:*

$$\lambda_j(\epsilon,n,\pm) = K_j^{\pm}\epsilon + o(\epsilon) \quad as \quad \epsilon \to 0$$

and there exists a positive constant μ_0, *independent of* $\epsilon \in (0,\epsilon_0]$ *such that:*

$$\lambda_{n+1}(\epsilon,n,\pm) \leq -\mu_0 \quad for \quad \epsilon \in (0,\epsilon_0].$$

(iii) *There exists a constant* d_1, $0 < d_1 \leq d_0$, *such that the functions* $u_{n,\pm}(x,\epsilon)$ *are monotonic over the intervals* $[x_j - \epsilon d_1, x_j + \epsilon d_1]$, $j = 1,\cdots,n$. *More precisely:*

$$\pm (-1)^{j+1} u'_{n,\pm}(x,\epsilon) > 0 \quad for \quad x \in [x_j - \epsilon d_1, x_j + \epsilon d_1]$$

$j = 1,2,\cdots,n.$

Proof: The proof is essentially the same as that of Theorem 4.3. We define $\phi_j(x,\epsilon)$ by: $\phi_j(x,\epsilon) = \sqrt{\epsilon}\, \bar{\Phi}_j(x,\epsilon)$, $j = 1,\cdots,n$ where $\bar{\Phi}_j$ are the j-th eigenfunctions of $\mathfrak{L}^{\epsilon,n}$, $j = 1,\cdots,n$, normalized in such a way as in (5.15). According to the decompositions in (5.20), the equilibrium solutions of (5.8) and (1.2) must satisfy:

$$(5.30) \quad \begin{cases} \text{a)} \quad \mathfrak{L}^{\epsilon,n} v + (I - E_n)G^n(\epsilon) + (I - E_n)F^n(\sum_{j=1}^{n} \alpha_j \phi_j(\epsilon) + v,\epsilon) = 0 \\ \text{b)} \quad \sum_{j=1}^{n} \alpha_j \lambda_j(\epsilon)\phi_j(\epsilon) + E_n G^n(\epsilon) + E_n F^n(\sum_{j=1}^{n} \alpha_j \phi_j(\epsilon) + v,\epsilon) = 0 \end{cases}$$

in which u is replaced by $\sum^{n} \alpha_j\phi_j(\epsilon) + v$, $\alpha_j \in \mathbb{R}$, $v \in X_n$. The equation (5.30-a) is uniquely solved in v as a function $v = v^*(\alpha,\epsilon)$ of (α,ϵ) in a neighborhood of $(\alpha,\epsilon) = (0,0) \in \mathbb{R}^n \times \mathbb{R}_+$. $v^*(\alpha,\epsilon)$ is continuous in (α,ϵ) and is as smooth in α as $f(u,x)$ is in u. Moreover,

$$|v^*(\alpha,\epsilon)|_{2,\epsilon} = O(\epsilon^2 + |\alpha|^2) \quad \text{as} \quad \epsilon,\ |\alpha| \to 0.$$

Substituting $v = v^*(\alpha,\epsilon)$ in (5.30-b), one obtains the bifurcation equations:

(5.31) $$B(\alpha,\epsilon) = 0,$$

where $B(\alpha,\epsilon) = (B_1(\alpha,\epsilon),\cdots,B_n(\alpha,\epsilon))$ is defined by

$$\begin{aligned} B_j(\alpha,\epsilon) &= \alpha_j\lambda_j(\epsilon) + \tfrac{1}{\epsilon}\langle G^n(\epsilon),\phi_j(\epsilon)\rangle \\ &\quad + \tfrac{1}{\epsilon}\langle F^n(\sum_{k=1}^{n}\alpha_k\phi_k(\epsilon) + v^*(\alpha,\epsilon),\epsilon),\phi_j(\epsilon)\rangle \end{aligned}$$

for $j = 1,\cdots,n$.

We shall show the solvability of (5.31) by computing first few coefficients of the Taylor expansion of $B(\alpha,\epsilon)$ in α. If we let $B_j(\alpha,\epsilon) = B_j^{(0)}(\epsilon) + \sum_{i=1}^{n} B_{ji}^{(1)}(\epsilon)\alpha_i + \sum_{i,k=1}^{n} B_{jik}^{(2)}(\epsilon)\alpha_i\alpha_k + O(|\alpha|^3)$ then, employing Lemma 5.1 - (ii), we can easily obtain:

$$\begin{aligned} B_j^{(0)}(\epsilon) &= \tfrac{1}{\epsilon}\langle G^n(\epsilon),\phi_j(\epsilon)\rangle + \tfrac{1}{\epsilon}\langle F^n(v^*(\cdot,\epsilon),\epsilon),\phi_j(\epsilon)\rangle \\ &= \tfrac{1}{\epsilon}\langle G^n(\epsilon),\phi_j(\epsilon)\rangle + O(\epsilon^4) \\ &= \epsilon^2\sum_{\ell=1}^{n} a_{j\ell}\tau_\ell + o(\epsilon^2) = o(\epsilon^2) \end{aligned}$$

where $$\begin{aligned} \tau_\ell &= \int_{-\infty}^{\infty} [\tfrac{1}{2} f_{uu}(z_0^\ell(t),x_\ell)z_1^\ell(t)^2 + f_{ux}(z_0^\ell(t),\ x_\ell)tz_1^\ell(t) \\ &\quad + \tfrac{1}{2} f_{xx}(z_0^\ell(t),x_\ell)t^2]\dot z_0^\ell(t)dt \\ &= 0 \quad \text{(by virtue of (5.5))} \end{aligned}$$

and $a_{j\ell}$ are the constants defined in (5.16). The same type of computation

shows

$$B_{jj}^{(1)}(\epsilon) = \lambda_j^{\pm}(\epsilon) + o(\epsilon) = K_j^{\pm}\epsilon + o(\epsilon), \; j = 1,\cdots,n$$

$$B_{ji}^{(1)}(\epsilon) = O(\epsilon^2) \qquad i \neq j,$$

$$B_{jik}^{(2)}(\epsilon) = \frac{1}{2}\sum_{\ell=1}^{n} a_{j\ell}a_{i\ell}a_{k\ell}\int_{-\infty}^{\infty} f_{uu}(z_0^{\ell}(t),x_{\ell})[\dot z_0^{\ell}(t)]^3 dt + o(1)$$

$$= o(1) \qquad\qquad j,i,k = 1,\cdots,n$$

in which $o(\epsilon)/_{\epsilon}$, $o(1) \to 0$ as $\epsilon \to 0$. Therefore, the bifurcation equations (5.31) read as the following:

$$o(\epsilon^2) + \epsilon\begin{bmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_n \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + N(\epsilon)\alpha + o(1)O(|\alpha|^2) + O(|\alpha|^3) = 0$$

in which $N(\epsilon)$ is a certain $n \times n$ matrix with zero diagonal entries and $|N(\epsilon)| = O(\epsilon^2)$. Now, applying the implicit function theorem to $\frac{1}{\epsilon^2}B(\bar\alpha\epsilon,\epsilon) = 0$, one obtains the solution of (5.31), $\alpha_j = \alpha_j^*(\epsilon)$, with $\alpha_j^*(\epsilon) = o(\epsilon)$, $j = 1,\cdots,n$. The desired families of equilirium solutions of (1.1), (1.2) are given by:

$$u_{n,\pm}(x,\epsilon) = U_{n,\pm}(x,\epsilon) + \sum_{j=1}^{n}\alpha_j^*(\epsilon)\phi_j^{\pm}(x,\epsilon) + v^*(\alpha^*(\epsilon),\epsilon).$$

The rest of the proof is nearly identical to that of Theorem 4.3. This completes the proof of Theorem 5.4.

Remark 5.5. Theorem 5.4 also implies that the equilibrium solutions $u_{n,\pm}(\cdot,\epsilon)$ are hyperbolic and that the index of them, $\mathrm{Ind}(u_{n,\pm}(\cdot,\epsilon)) := \dim W^u(u_{n,\pm}(\cdot,\epsilon))$, is obtained by looking at the signs of $K_j^{\pm}$, $j = 1,\cdots,n$. Namely,

(5.2) $$\mathrm{Ind}(u_{n,\pm}(\cdot,\epsilon)) = \mathrm{Card}\{j \,|\, K_j^{\pm} > 0\}$$

and from $(5.13)_{\pm}$ it easily follows that:

(5.33) $$\mathrm{Ind}(u_{n,+}(\cdot,\epsilon)) + \mathrm{Ind}(u_{n,-}(\cdot,\epsilon)) = n.$$

6. Generalizations.

In this section, we generalize the results in previous sections for the following parabolic equation

(6.1) $\partial u/\partial t = \epsilon^2(a(x)u')' + f(u,x), \ x \in [-1,1], \ t \geq 0$

subjected to the Neumann boundary conditions

(6.2) $u'(-1,t) = 0 = u'(1,t).$

The functions a and f will satisfy the conditions:

(B-1) $a : [-1,1] \to \mathbb{R}$ is C^∞-function of x

$f = \mathbb{R} \times [-1,1] \to \mathbb{R}$ is a C^∞-function of (u,x)

and there exist two functions $h_0(x)$ and $h_1(x)$ of class C^∞, and a constant $a_0 > 0$ such that

(i) $a(x) \geq a_0$,

(ii) $f(h_i(x),x) = 0, \ x \in [-1,1], \ i = 0,1$

(iii) $h_i'(\pm 1) = 0, \ i = 0,1$

(B-2) There exists a positive constant β such that

$f_u(h_i(x),x) \leq -3\beta^2, \ x \in [-1,1], \ i = 0,1.$

(B-3) If we define $J(x) = \int_{h_0(t)}^{h_1(x)} f(s,x)ds, \ x \in [-1,1]$ then there exist n points $x_j \in (-1,1), \ x_j < x_{j+1}, j = 1,\cdots,n-1$ such that:

(i) $J(x_j) = 0, \ j = 1,\cdots,n$

(ii) $J'(x_j) \neq 0, \ j = 1,\cdots,n$

(iii) $\int_{h_0(x_j)}^{u} f(s,x_j)ds < 0$ for u in the open interval between $h_0(x_j)$ and $h_1(x_j), \ j = 1,\cdots,n.$

Under the hypotheses (B-1) ~ (B-3), it is natural to expect that the same type of results as in Theorem 5.4 will be true for the problem (6.1), (6.2). In fact, we shall show the following.

Theorem 6.1. *Suppose that the conditions* (B-1), (B-2) *and* (B-3) *are satisfied. Then, there exist a constant* $\epsilon_0 > 0$ *and two families of equilibrium solutions of* (6.1), (6.2), $u_{n,\pm}$, *for which the statements* (i), (ii) *and* (iii) *of Theorem* 5.4 *are valid, with the statement* (i) *replaced by*

(i) $$\lim_{\epsilon\to 0} u_{n,+}(x,\epsilon) = \begin{Bmatrix} h_0(x) & \text{on} & \Omega_0 \\ h_1(x) & \text{on} & \Omega_1 \end{Bmatrix} \text{ compact uniformly}$$

$$\lim_{\epsilon\to 0} u_{n,-}(x,\epsilon) = \begin{Bmatrix} h_1(x) & \text{on} & \Omega_0 \\ h_0(x) & \text{on} & \Omega_1 \end{Bmatrix} \text{ compact uniformly}.$$

In order to prove this result, we need the following

Theorem 6.2. *Under the assumptions* (B-1) *and* (B-2), *there exist a constant* $\epsilon_1 > 0$ *and two families of equilibrium solutions* $w_i(x,\epsilon)$, $i = 0,1$, *of* (6.1), (6.2), *defined for* $\epsilon \in (0,\epsilon_1]$ *such that:*

(i) $|w_i(\cdot,\epsilon) - h_i(\cdot)|_2 = O(\epsilon^2)$ *as* $\epsilon \to 0$, $i = 0,1$

(ii) $w_i(\cdot,\epsilon)$, $i = 0,1$, $\epsilon \in (0,\epsilon_1]$ *are asymptotically stable solutions of* (6.1), (6.2).

(iii) *The families* $w_i(\cdot,\epsilon)$, $i = 0,1$, *are unique with respect to the property* : $|w_i(\cdot,\epsilon) - h_i(\cdot)|_0 \to 0$ *as* $\epsilon \to 0$.

(iv) *If we define* $w_i(x,0) = h_i(x)$, $i = 0,1$, *then the functions* $w_i(x,\epsilon)$, $\epsilon w_i'(x,\epsilon)$ *and* $\epsilon^2 w_i''(x,\epsilon)$, $i = 0,1$, *are* C^2*-functions of* (x,ϵ) *on* $[-1,1] \times [0,\epsilon]$.

Since the proof of this theorem is identical for both $i = 0$ and $i = 1$, the subscript i will be suppressed in the sequel.

By the change of variables $u = v + h(x)$ in (6.1), (6.2), the new function v is subject to the equation

$$\partial v/\partial t = A(\epsilon)v + G(\epsilon) + F(v) \tag{6.3}$$

where $A(\epsilon) : X \to Y$, $G(\epsilon) \in Y$, $F(\cdot) : Y \to Y$ are given by

$$A(\epsilon)u = \epsilon^2(a(x)v')' + f_u(h(x),x)v$$

$$G(\epsilon)(x) = \epsilon^2(a(x)h'(x))'.$$

$$F(v)(x) = f(h(x) + v(x),x) - f_u(h(x),x)v(x).$$

One should notice that $|F(v)|_0 = O(|v|_0^2)$ as $|v|_0 \to 0$ since $f(h(x),x) \equiv 0$.

Lemma 6.3. (i) *The linear operator* $A(\epsilon) : X_\epsilon \to Y$ *is invertible with the inverse bounded uniformly with respect to* $\epsilon \in (0,\epsilon_1]$ *for some* $\epsilon_1 > 0$, *namely, there exists a constant* $c > 0$ *such that*

$$\|A(\epsilon)^{-1}\|_{B(Y,X_\epsilon)} \leq c, \quad \textit{for} \quad \epsilon \in (0,\epsilon_1].$$

(ii) *The eigenvalues of* $A(\epsilon)$ *are contained in* $(-\infty,-2\beta^2]$ *for* $\epsilon \in (0,\epsilon_1]$.

Proof. Since $f_u(h(x),x) \leq -3\beta^2$, the linear analysis in §3 implies the existence of such a constant c as above. In fact, the equations for the eigenvalue problem $A(\epsilon)v = \lambda v$, are given by

$$\begin{cases} \epsilon r' = -r[1/a(x)\cdot + \lambda - f_u(h(x),x)] \sin\theta \cos\theta \\ \epsilon\theta' = [f_u(h(x),x) - \lambda] \cos^2\theta + \sin^2\theta/a(x) \end{cases}$$

in terms of the polar coordinates (r,θ) defined by $v = r\cos\theta$, $v' = -r\sin\theta$. The equation for the angle θ shows that the first eigenvalue of $A(\epsilon)$, for $\epsilon \in [0,\epsilon_1]$, with $\epsilon_1 > 0$ small enough, lies in $(-\infty, -2\beta^2]$, which together with the self-adjointness of $A(\epsilon)$ gives

$$\|A(\epsilon)^{-1}\|_{B(Y,Y)} \leq {}^1/2\beta^2.$$

The same type of argument as in the proof of Lemma 4.1 completes the proof of Lemma 6.3.

Proof of Theorem 6.2. The equilibrium solutions of (6.3) are a fixed point of the operator $\mathcal{F}(\cdot,\epsilon) = Y \to Y$, which is defined by

$$\mathcal{F}(v,\epsilon) = -A(\epsilon)^{-1}[G(\epsilon) + F(v)].$$

Since $|G(\epsilon)|_0 = O(\epsilon^2)$ as $\epsilon \to 0$ and $|F(v)|_0 = O(|v|_0^2)$ as $|v|_0 \to 0$, Lemma 6.3 implies that there exists a constant $c > 0$ such that

$$\text{(6.4)} \qquad \begin{aligned} &|\mathcal{F}(v,\epsilon)|_{2,\epsilon} \leq c[\epsilon^2 + |v|_0^2],\ v \in Y. \\ &|\mathcal{F}(v_1,\epsilon) - \mathcal{F}(v_2,\epsilon)|_{2,\epsilon} \leq c\{|v_1|_0 + |v_2|_0\}|v_1 - v_2|_0. \end{aligned}$$

If we let $Y(r) = \{v \in Y;\ |v|_0 \leq r\}$, and if we choose $\epsilon_1 > 0$, $r > 0$ so small that the inequalities

$$c[\epsilon_1^2 + r^2] < r, \quad \text{and} \quad cr < \tfrac{1}{2}$$

hold, then the mapping $F(\cdot,\epsilon) : Y(r) \to Y(r)$ is a contraction mapping on the complete metric space $Y(r)$. Therefore, there exists a unique fixed point $u(\cdot,\epsilon)$ of $F(\cdot,\epsilon)$ in $Y(r)$ for each $\epsilon \in (0,\epsilon_1]$. This proves the existence of the desired family $w(x,\epsilon) = h(x) + v(x,\epsilon)$ and its uniqueness (part (iii)). Since $v(\cdot,\epsilon)$ is a fixed point of $F(\cdot,\epsilon)$, the estimate (6.4) implies $|v(\cdot,\epsilon)|_{2,\epsilon} \leq c'\epsilon^2$ for some $c' > 0$, and in particular, $|v(\cdot,\epsilon)|_0 \leq c'\epsilon^2$. Therefore, in view of the proof of Lemma 6.3, part (ii) follows immediately. Parts (i) and (iv) will be proved by a kind of bootstrap argument. First of all, one should notice that the function $v(x,\epsilon)$ is a smooth function of (x,ϵ) on $[-1,1] \times (0,\epsilon_1]$. Now, the equation for $v(x,\epsilon)$ is given by

$$\epsilon^2(a(x)v')' + f(h(x) + v,x) = 0.$$

Differentiating this expression with respect to x, one obtains:

$$\tilde{A}(\epsilon)V + p(x,\epsilon) = 0 \tag{6.5}$$

in which $p(x,\epsilon) = a'(x)\epsilon^2 v''(x,\epsilon) + a''(x)\epsilon^2 v'(x,\epsilon) + f_u(h(x) + v(x,\epsilon),x)h'(x) + f_x(h(x) + v(x,\epsilon),x)$ and $V(x,\epsilon) = \partial v(x,\epsilon)/\partial x$ and $\tilde{A}(\epsilon)V \equiv \epsilon^2(a(x)V')' + f_u(w(x,\epsilon),x)V$. Since Lemma 6.3 also applies to $\tilde{A}(\epsilon)$, for $\epsilon \in (0,\epsilon_1]$ by reducing $\epsilon_1 > 0$ if necessary, (6.5) implies

$$|V|_{2,\epsilon} \leq c|p(\cdot,\epsilon)|_0. \tag{6.6}$$

Since $|v(\cdot,\epsilon)|_{2,\epsilon} = O(\epsilon^2)$, and $f_u(h(x),x)h'(x) + f_x(h(x),x) \equiv 0$ imply $|p(\cdot,\epsilon)|_{2,\epsilon} \leq c\epsilon^2$ it follows from (6.6) that $|v'(\cdot,\epsilon)|_{2,\epsilon} \leq c\epsilon^2$. Differentiating (6.5) again with respect to x, one obtains:

$$\tilde{A}(\epsilon)W + q(x,\epsilon) = 0 \tag{6.7}$$

where $W(x,\epsilon) = v''(x,\epsilon)$ and $|q(\cdot,\epsilon)|_0 = O(\epsilon^2)$. In order to prove

$|q(\cdot,\epsilon)|_0 = O(\epsilon^2)$, we use two facts: 1) $|v|_{2,\epsilon} = O(\epsilon^2)$.
2) $f_{uu}(h(x),x)[h'(x)]^2 + 2f_{ux}(h(x),x)h'(x) + f_{xx}(h(x),x) + f_u(h(x),x)h''(x) \equiv 0$. Hence, employing Lemma 6.3 again, one obtains

$$|W(\cdot,\epsilon)|_{2,\epsilon} \leq c\,|q(\cdot,\epsilon)|_0 = c\,\epsilon^2$$

and in particular $|v''(\cdot,\epsilon)|_0 \leq c\epsilon^2$. Therefore, $|w(\cdot,\epsilon) - h(\cdot)|_2 \leq c\epsilon^2$, which proves part (i). Part (iv) follows from part (i) and the estimate $|W(\cdot,\epsilon)|_{2,\epsilon} \leq c\epsilon^2$. This completes the proof of Theorem 6.2.

We now proceed to:

Proof of Theorem 6.1. By using the functions $w_i(x,\epsilon)$, $i = 0,1$ in Theorem 6.2, let us define:

$$W(x,\epsilon) = w_1(x,\epsilon) - w_0(x,\epsilon).$$

We can assume that $|W(x,\epsilon)| \geq M > 0$, $\epsilon \in [0,\epsilon_1]$ for some positive constant M. If we change variables in (6.1) by $u \longmapsto W(x,\epsilon)u + w_0(x,\epsilon)$, and multiply the result by $W(x,\epsilon)$, then the new function u is subject to the equation

$$W(x,\epsilon)^2\,\partial u/\partial t = \epsilon^2(\tilde{a}(x,\epsilon)u')' + \tilde{f}(u,x,\epsilon) \tag{6.8}$$

and the boundary conditions in (6.2), where the functions $\tilde{a}(x,\epsilon)$ and $\tilde{f}(u,x,\epsilon)$ are given by:

$$\tilde{a}(x,\epsilon) = a(x)W(x,\epsilon)^2$$
$$\tilde{f}(u,x,\epsilon) = W(x,\epsilon)[\epsilon^2(a(x)W'(x,\epsilon))' + \epsilon^2(a(x)w_0'(x))' + f(W(x,\epsilon)u + w_0(x,\epsilon),x)].$$

These functions satisfy the conditions below:

a) $\tilde{a}(\cdot,\cdot) = [-1,1] \times [0,\epsilon_1] \to \mathbb{R}$ is C^2 in (x,ϵ)

and

$\tilde{f}(\cdot,\cdot,\cdot) : \mathbb{R} \times [-1,1] \times [0,\epsilon_1] \to \mathbb{R}$ is C^∞ in u and C^2 in (x,ϵ).

b) There exists a positive constant $\tilde{a}_0$ such that

$$\tilde{a}(x,\epsilon) \geq \tilde{a}_0,\ (x,\epsilon) \in [-1,1] \times [0,\epsilon_1]$$

c) $\tilde{f}(i,x,\epsilon) \equiv 0,\ i = 0,1,\ \epsilon \in [0,\epsilon_1]$.

d) There exists a positive constant $\bar{\beta}$ such that

$$\tilde{f}_u(i,x,\epsilon) \leq -3\bar{\beta}^2,\ (x,\epsilon) \in [-1,1] \times [0,\epsilon_1],\ i = 0,1.$$

e) If $\tilde{J}(x)$ is defined by $\tilde{J}(x) = \int_0^1 \tilde{f}(s,x,0)ds$, then, the conditions for $J(x)$ in (B-3)' remain satisfied for $\tilde{J}(x)$ and $\int_0^u \tilde{f}(x,s,0)ds < 0$ for $u \in (0,1)$.

In fact a) is the consequence of Theorem 6.2 and b), c) and d) follow from the conditions (B-1) and (B-2) with the fact that $w_i(x,\epsilon)$, $i = 0,1$, are equilibrium solutions of (6.1), (6.2). As for the property e), it suffices to notice:

$$\begin{aligned}\tilde{J}(x) &= W(x,0)\int_0^1 f(W(x,0)u + w_0(x,0),\ x)du \\ &= \int_{h_0(x)}^{h_1(x)} f(s,x)ds = J(x).\end{aligned}$$

The conditions a) through e) above are sufficent for the procedures in §2 through §5 to work. Then, transforming back to the original variables by $u \longmapsto [u - w_0(x,\epsilon)]/W(x,\epsilon)$ one can complete the proof of Theorem 6.1.

Remark 6.4. Theorem 6.2 was previously proved by Fife [1974]. Our proof is different from his in that we are free from the maximal principle in order to obtain uniform invertibility of the linear operator $A(\epsilon)$.

Remark 6.5. One cannot apply the procedures up through §5 directly to prove Theorem 6.1. The difference between the problems in §5 and §6 lies in that $u = 0,1$ are equilibrium solutions of (1.1), (1.2), but $h_i(x)$, $i = 0,1$, are not equilibrium solutions of (6.1), (6.2). Theorem 6.2 plays a role to bridge the gap between them.

Remark 6.6. Theorem 6.1 could be more generalized. For instance, it still remains true when $h_i(x)$, $i = 0,1$ are defined on the unions of subintervals, say $\Omega_i = \bigcup_{j=1}^{n_i} I_{ij}$, $i = 0,1$, $\Omega_0 \cup \Omega_1 = [-1,1]$ with any two adjacent subintervals I_{0i} and I_{1j} being overlapped.

Remark 6.7. The idea developed in the present paper may prove useful in order to show the existence of transition layers and their stability for equations in several space dimensions.

Remark 6.8. The methods presented in Sections 2 - 6 apply to show the existence of Neumann boundary layers and interior double layers. Specific feature of these types of solutions is that they are unstable. Boundary layers turn out to be rather easy to handle because the linearized operator $\mathfrak{L}^\epsilon$ around approximate solutions does not have small eigenvalues approaching zero as $\epsilon \to 0$.

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